Mathematical Tables
and other
Aids to Computation

A Quarterly Journal

Edited by

C. C. CRAIG ALAN FLETCHER EUGENE ISAACSON

C. V. L. SMITH A. H. TAUB

C. B. TOMPKINS, *Chairman*

XII

NOS. 61–64

1958

Published by

THE NATIONAL RESEARCH COUNCIL

Washington, D. C.

Mathematics

QA
47
.M43
V. 12
1958

Volume XII January 1958 Number 61

UNIVERSITY OF MICHIGAN
DEC 9 1958
MATHEMATICS LIBRARY

Mathematical Tables

and other

Aids to Computation

Published Quarterly
by the
National Academy of Sciences—National Research Council

Published quarterly in January, April, July, and October by the National Academy of Sciences—National Research Council, Prince and Lemon Sts., Lancaster, Pa., and Washington, D. C.

Entered as second-class matter July 20, 1943, at the post office at Lancaster, Pennsylvania, under the Act of August 24, 1912.

Editorial Committee

Division of Mathematics

National Academy of Sciences—National Research Council

Washington, D. C.

C. B. TOMPKINS, *Chairman*, University of California, Los Angeles, California

C. C. CRAIG, University of Michigan, Ann Arbor, Michigan

ALAN FLETCHER, University of Liverpool, Liverpool 3, England

EUGENE ISAACSON, Institute of Mathematical Sciences, New York University, New York 3, New York

C. V. L. SMITH, Ballistic Research Labs., Aberdeen Proving Ground, Aberdeen, Maryland

A. H. TAUB, University of Illinois, Urbana, Illinois

• • •

Information to Subscribers

The journal is published quarterly in one volume per year with issues numbered serially since Volume I, Number 1. Subscriptions are $5.00 per year, single copies $1.50. Back issues are available as follows:

Volume I (1943–1945), Nos. 10 and 12 *only* are available; $1.00 per issue.

Volume II (1946–1947), Nos. 13, 14, 17, 18, 19, and 20 *only* are available; $1.00 per issue

Volume III (1948–1949), Nos. 21–28 available; $4.00 per year (four issues), $1.25 per issue

Volume IV (1950 and following), all issues available; $5.00 per year, $1.50 per issue

All payments are to be made to the National Academy of Sciences and forwarded to the Publications Office, 2101 Constitution Avenue, Washington, D. C.

Agents for Great Britain and Ireland: Scientific Computing Service, Ltd., 23 Bedford Square, London W.C.1

• • •

Microcard Edition

Volumes I–X (1943–1956), Nos. 1–56 are now available on Microcards and may be purchased from The Microcard Foundation, Box 2145, Madison 5, Wisconsin, at a cost of $20.00 for the complete set. Future volumes will be available on Microcards in the year following original publication.

• • •

Information to Contributors

All contributions intended for publication in *Mathematical Tables and Other Aids to Computation* and all books for review should be addressed to C. B. Tompkins, Department of Mathematics, University of California, Los Angeles 24, California. The author should mention the name of an appropriate editor for his paper if this is convenient.

2564
8841A

Coefficients and Roots of the Polynomials which Define the Derivatives of the Exponential of $(-\epsilon/T)$

By Edwin S. Campbell, E. M. Fishbach, and J. O. Hirschfelder

The n-th derivative of $\exp(-\epsilon/T)$ can be shown to be of the form

$$d^n \exp[-\epsilon/T]/dT^n = T^{-n} \exp[-\epsilon/T] W_n[\epsilon/T] \tag{1}$$

$$W_n[\epsilon/T] = \sum_{l=1}^{n} b_{n,l}[\epsilon/T]^l.$$

The $b_{n,l}$ are constant coefficients in a homogeneous polynomial of the n-th order in $[\epsilon/T]$. These polynomials attain added importance from their relation to the Laguerre polynomials [1]

$$W_n[\epsilon/T] = (-1)^n\{L_n[\epsilon/T] - nL_{n-1}[\epsilon/T]\} \tag{2}$$

$L_n[\epsilon/T]$: the n-th order Laguerre polynomial.

The zeros of these polynomials are of interest since they locate the extreme values of the derivatives of $\exp(-\epsilon/T)$. Furthermore, an accurate, simple computation of the W_n for values of (ϵ/T) which occur in physical problems sometimes requires use of the polynomial roots. This occurs whenever the direct evaluation of $W_n[\epsilon/T]$ by synthetic division introduces at intermediate steps of the calculation numbers which are larger than the value of the polynomial. In such cases the subtraction of these larger numbers can require the use of considerably more than the usual guard figures which allow for the effect of ordinary round-off error. Fortunately the occurrence of this disastrous subtraction in synthetic division does not imply the occurrence when the polynomial is calculated using its roots

$$W_n[\epsilon/T] = \prod_{l=1}^{n} [\epsilon/T - r_{n,l}] \tag{3}$$

$r_{n,l}$: the l-th root of W_n.

The polynomial coefficients, $b_{n,l}$, are integers. Table I lists all of their non-zero digits which were computed from the recursion relations

$$b_{n,1} = (-1)^{n-1}n! \tag{4}$$

$$b_{n,l} = -(n + l - 1)b_{n-1,l} + b_{n-1,l-1}(2 \leq l \leq n - 1)$$

$$b_{n,n} = 1.$$

The values in the Table were checked numerically by the distinct relations

$$(n - l)b_{n,l} = -n(n - 1)b_{n-1,l}. \tag{5}$$

Received 16 September 1957.

TABLE I. *Exact Values of the Coefficients,* $b_{n,l}$

n	l	$\beta_{n,l}$	$P_{n,l}$
1	1	+1	0
2	1	−2	0
	2	+1	0
3	1	+6	0
	2	−6	0
	3	+1	0
4	1	−2.4	1
	2	+3.6	1
	3	−1.2	1
	4	+1	0
5	1	+1.2	2
	2	−2.4	2
	3	+1.2	2
	4	−2	1
	5	+1	0
6	1	−7.2	2
	2	+1.8	3
	3	−1.2	3
	4	+3	2
	5	−3	1
	6	+1	0
7	1	+5.04	3
	2	−1.512	4
	3	+1.26	4
	4	−4.2	3
	5	+6.3	2
	6	−4.2	1
	7	+1	0
8	1	−4.032	4
	2	+1.4112	5
	3	−1.4112	5
	4	+5.88	4
	5	−1.176	4
	6	+1.176	3
	7	−5.6	1
	8	+1	0
9	1	+3.6288	5
	2	−1.4515 2	6
	3	+1.6934 4	6
	4	−8.4672	5
	5	+2.1168	5
	6	−2.8224	3
	7	+2.016	1
	8	−7.2	0
	9	+1	4
10	1	−3.6288	6
	2	+1.6329 6	7
	3	−2.1772 8	7
	4	+1.2700 8	7
	5	−3.8102 4	6
	6	+6.3504	5
	7	−6.048	4
	8	+3.24	3
	9	−9	1
	10	+1	0
11	1	+3.9916 8	7
	2	−1.9958 4	8
	3	+2.9937 6	8
	4	−1.9958 4	8
	5	+6.9854 4	7
	6	−1.3970 88	7
	7	+1.6632	6
	8	−1.188	5
	9	+4.95	3
	10	−1.1	2
	11	+1	0
12	1	−4.7900 16	8
	2	+2.6345 088	9
	3	−4.3908 48	9
	4	+3.2931 36	9
	5	−1.3172 544	9
	6	+3.0735 936	8
	7	−4.3908 48	7
	8	+3.9204	6
	9	−2.178	5
	10	+7.26	3
	11	−1.32	2
	12	+1	0
13	1	+6.2270 208	9
	2	−3.7362 1248	10
	3	+6.8497 2288	10
	4	−5.7081 024	10
	5	+2.5686 4608	10
	6	−6.8497 2288	9
	7	+1.1416 2048	9
	8	−1.2231 648	8
	9	+8.4942	6
	10	−3.7752	5
	11	+1.0296	4
	12	−1.56	2
	13	+1	0
14	1	−8.7178 2912	10
	2	+5.6665 88928	11
	3	−1.1333 17785 6	12
	4	+1.0388 74636 8	12
	5	−5.1943 73184	11
	6	+1.5583 11955 2	11
	7	−2.9682 13248	10
	8	+3.7102 6656	9
	9	−3.0918 888	8
	10	+1.7177 16	7
	11	−6.2462 4	5
	12	+1.4196	4
	13	−1.82	2
	14	+1	0
15	1	+1.3076 74368	12
	2	−9.1537 20576	12
	3	+1.9833 06124 8	13
	4	−1.9833 06124 8	13
	5	+1.0908 18368 64	13
	6	−3.6360 61228 8	12
	7	+7.7915 59776	11
	8	−1.1130 79968	11
	9	+1.0821 6108	10
	10	−7.2144 072	8
	11	+3.2792 76	7
	12	−9.9372	5
	13	+1.911	4
	14	−2.1	2
	15	+1	0

TABLE I. *Exact Values of the Coefficients, $b_{n,l}$*—Continued

n	l	$\beta_{n,l}$	$P_{n,l}$
16	1	−2.0922 78988 8	13
	2	+1.5692 09241 6	14
	3	−3.6614 88230 4	14
	4	+3.9666 12249 6	14
	5	−2.3799 67349 76	14
	6	+8.7265 46949 12	13
	7	−2.0777 49273 6	13
	8	+3.3392 39904	12
	9	−3.7102 6656	11
	10	+2.8857 6288	10
	11	−1.5740 5248	9
	12	+5.9623 2	7
	13	−1.5288	6
	14	+2.52	4
	15	−2.4	2
	16	+1	0
17	1	+3.5568 74280 96	14
	2	−2.8454 99424 768	15
	3	+7.1137 48561 92	15
	4	−8.2993 73322 24	15
	5	+5.3945 92659 456	15
	6	−2.1578 37063 7824	15
	7	+5.6514 78024 192	14
	8	−1.0091 92504 32	14
	9	+1.2614 90630 4	13
	10	−1.1213 25004 8	12
	11	+7.1357 04576	10
	12	−3.2435 0208	9
	13	+1.0395 84	8
	14	−2.2848	6
	15	+3.264	4
	16	−2.72	2
	17	+1	0
18	1	−6.4023 73705 728	15
	2	+5.4420 17649 8688	16
	3	−1.4512 04706 63168	17
	4	+1.8140 05883 2896	17
	5	−1.2698 04118 30272	17
	6	+5.5024 84512 64512	16
	7	−1.5721 38432 18432	16
	8	+3.0881 29063 2192	15
	9	−4.2890 68143 36	14
	10	+4.2890 68143 36	13
	11	−3.1193 22286 08	12
	12	+1.6541 86060 8	11
	13	−6.3622 5408	9
	14	+1.7478 72	8
	15	−3.3292 8	6
	16	+4.1616	4
	17	−3.06	2
	18	+1	0
19	1	+1.2164 51004 08832	17
	2	−1.0948 05903 67948 8	18
	3	+3.1019 50060 42521 6	18
	4	−4.1359 33413 90028 8	18
	5	+3.1019 50060 42521 6	18
	6	−1.4475 76694 86510 08	18
	7	+4.4805 94531 72531 2	17
	8	−9.6012 73996 55424	16
	9	+1.4668 61305 02912	16
	10	−1.6298 45894 4768	15
	11	+1.3335 10277 2992	14
	12	−8.0818 80468 48	12
	13	+3.6264 84825 6	11
	14	−1.1955 44448	10
	15	+2.8465 344	8
	16	−4.7442 24	6
	17	+5.2326	4
	18	−3.42	2
	19	+1	0
20	1	−2.4329 02008 17664	18
	2	+2.3112 56907 76780 8	19
	3	−6.9337 70723 30342 4	19
	4	+9.8228 41858 01318 4	19
	5	−7.8582 73486 41054 72	19
	6	+3.9291 36743 20527 36	19
	7	−1.3097 12247 73509 12	19
	8	+3.0404 03432 24217 6	18
	9	−5.0673 39053 73696	17
	10	+6.1934 14399 01184	16
	11	−5.6303 76726 3744	15
	12	+3.8388 93222 528	14
	13	−1.9686 63191 04	13
	14	+7.5717 81504	11
	15	−2.1633 66144	10
	16	+4.5070 128	8
	17	−6.6279 6	6
	18	+6.498	4
	19	−3.8	2
	20	+1	0
21	1	+5.1090 94217 17094 4	19
	2	−5.1090 94217 17094 4	20
	3	+1.6178 79835 43746 56	21
	4	−2.4268 19753 15619 84	21
	5	+2.0627 96790 18276 864	21
	6	−1.1001 58288 09747 6608	21
	7	+3.9291 36743 20527 36	20
	8	−9.8228 41858 01318 4	19
	9	+1.7735 68668 80793 6	19
	10	−2.3647 58225 07724 8	18
	11	+2.3647 58225 07724 8	17
	12	−1.7914 83503 8464	16
	13	+1.0335 48175 296	15
	14	−4.5430 68902 4	13
	15	+1.5143 56300 8	12
	16	−3.7858 90752	10
	17	+6.9593 58	8
	18	−9.0972	6
	19	+7.98	4
	20	−4.2	2
	21	+1	0

TABLE I. *Exact Values of the Coefficients, $b_{n,l}$*—Continued

n	l	$\beta_{n,l}$	$P_{n,l}$
22	1	−1.1240 00727 77760 768	21
	2	+1.1802 00764 16648 8064	22
	3	−3.9340 02547 22162 688	22
	4	+6.2288 37366 43424 256	22
	5	−5.6059 53629 79081 8304	22
	6	+3.1767 07056 88146 37056	22
	7	−1.2101 74116 90722 42688	22
	8	+3.2415 37813 14435 072	21
	9	−6.3029 90192 22512 64	20
	10	+9.1043 19166 54740 48	19
	11	−9.9319 84545 32444 16	18
	12	+8.2766 53787 77036 8	17
	13	−5.3055 47299 8528	16
	14	+2.6236 22291 136	15
	15	−9.9947 51585 28	13
	16	+2.9151 35879 04	12
	17	−6.4304 46792	10
	18	+1.0507 266	9
	19	−1.2289 2	7
	20	+9.702	4
	21	−4.62	2
	22	+1	0
23	1	+2.5852 01673 88849 7664	22
	2	−2.8437 21841 27734 74304	23
	3	+9.9530 26444 47071 60064	23
	4	−1.6588 37740 74511 93344	24
	5	+1.5758 95853 70786 33676 8	24
	6	−9.4553 75122 24718 02060 8	23
	7	+3.8271 75644 71909 67500 8	23
	8	−1.0934 78755 63402 76428 8	23
	9	+2.2780 80740 90422 4256	22
	10	−3.5436 81152 51768 2176	21
	11	+4.1879 86816 61180 6208	20
	12	−3.8072 60742 37436 928	19
	13	+2.6846 06933 72551 68	18
	14	−1.4750 58754 79424	17
	15	+6.3216 80377 6896	15
	16	−2.1072 26792 5632	14
	17	+5.4230 10127 92	12
	18	−1.0633 35319 2	11
	19	+1.5545 838	9
	20	−1.6364 04	7
	21	+1.1688 6	5
	22	−5.06	2
	23	+1	0
24	1	−6.2044 84017 33239 43936	23
	2	+7.1351 56619 93225 35526 4	24
	3	−2.6162 24093 97515 96359 68	25
	4	+4.5783 92164 45652 93629 44	25
	5	−4.5783 92164 45652 93629 44	25
	6	+2.8996 48370 82246 85965 312	25
	7	−1.2427 06444 63820 08270 848	25
	8	+3.7725 01706 93739 53679 36	24
	9	−8.3833 37126 52754 52620 8	23
	10	+1.3972 22854 42125 75436 8	23
	11	−1.7782 83632 89978 23283 2	22
	12	+1.7513 39941 49220 98688	21
	13	−1.3471 84570 37862 2976	20
	14	+8.1423 24326 46420 48	18
	15	−3.8772 97298 31628 8	17
	16	+1.4539 86486 86860 8	16
	17	−4.2764 30843 7312	14
	18	+9.7826 84936 64	12
	19	−1.7162 60515 2	11
	20	+2.2582 3752	9

TABLE I. *Exact Values of the Coefficients,* $b_{n,l}$—Continued

n	l	$\beta_{n,l}$	$P_{n,l}$
24	21	−2.1507 024	7
	22	+1.3965 6	5
	23	−5.52	2
	24	+1	0
25	1	+1.5511 21004 33309 85984	25
	2	−1.8613 45205 19971 83180 8	26
	3	+7.1351 56619 93225 35526 4	26
	4	−1.3081 12046 98757 98179 84	27
	5	+1.3735 17649 33695 88088 832	27
	6	−9.1567 84328 91305 87258 88	26
	7	+4.1423 54815 46066 94236 16	26
	8	−1.3314 71190 68378 66004 48	26
	9	+3.1437 51422 44782 94732 8	25
	10	−5.5888 91417 68503 01747 2	24
	11	+7.6212 15569 57049 56928	23
	12	−8.0831 07422 27173 7856	22
	13	+6.7359 22851 89311 488	21
	14	−4.4412 67814 43502 08	20
	15	+2.3263 78378 98977 28	19
	16	−9.6932 43245 79072	17
	17	+3.2073 23132 7984	16
	18	−8.3851 58517 12	14
	19	+1.7162 60515 2	13
	20	−2.7098 85024	11
	21	+3.2260 536	9
	22	−2.7931 2	7
	23	+1.656	5
	24	−6	2
	25	+1	0
26	1	−4.0329 14611 26605 63558 4	26
	2	+5.0411 43264 08257 04448	27
	3	−2.0164 57305 63302 81779 2	28
	4	+3.8648 76502 46330 40076 8	28
	5	−4.2513 64152 70963 44084 48	28
	6	+2.9759 54906 89674 40859 136	28
	7	−1.4171 21384 23654 48028 16	28
	8	+4.8080 90410 80256 27238 4	27
	9	−1.2020 22602 70064 06809 6	27
	10	+2.2704 87138 43454 35084 8	26
	11	−3.3025 26746 81388 14668 8	25
	12	+3.7528 71303 19759 2576	24
	13	−3.3679 61425 94655 744	23
	14	+2.4056 86732 81896 96	22
	15	−1.3746 78133 03941 12	21
	16	+6.3006 08109 76396 8	19
	17	−2.3164 00040 3544	18
	18	+6.8129 41295 16	16
	19	−1.5936 70478 4	15
	20	+2.9357 08776	13
	21	−4.1938 6968	11
	22	+4.5388 2	9
	23	−3.588	7
	24	+1.95	5
	25	−6.5	2
	26	+1	0
27	1	+1.0888 86945 04183 52160 768	28
	2	−1.4155 53028 55438 57808 9984	29
	3	+5.8981 37618 97660 74204 16	29
	4	−1.1796 27523 79532 14840 832	30
	5	+1.3565 71652 36461 97066 9568	30
	6	−9.9481 92117 34054 45157 6832	29
	7	+4.9740 96058 67027 22578 8416	29
	8	−1.7764 62878 09652 58063 872	29
	9	+4.6878 88150 53249 86557 44	28

TABLE I. *Exact Values of the Coefficients, $b_{n,l}$*—Continued

n	l	$\beta_{n,l}$	$P_{n,l}$
27	10	−9.3757 76301 06499 73114 88	27
	11	+1.4489 83610 16459 04935 936	27
	12	−1.7563 43769 89647 33255 68	26
	13	+1.6887 92086 43891 66592	25
	14	−1.2990 70835 72224 3584	24
	15	+8.0418 67078 28055 552	22
	16	−4.0209 33539 14027 776	21
	17	+1.6261 12828 32878 88	20
	18	−5.3140 94210 2248	18
	19	+1.3984 45844 796	17
	20	−2.9440 96515 36	15
	21	+4.9068 27525 6	13
	22	−6.3725 0328	11
	23	+6.2969 4	9
	24	−4.563	7
	25	+2.2815	5
	26	−7.02	2
	27	+1	0
28	1	−3.0488 83446 11713 86050 1504	29
	2	+4.1159 92652 25813 71167 70304	30
	3	−1.7835 96815 97852 60839 33798 4	31
	4	+3.7158 26699 95526 26748 6208	31
	5	−4.4589 92039 94631 52098 34496	31
	6	+3.4185 60563 95884 16608 73113 6	31
	7	−1.7906 74581 12129 80128 38297 6	31
	8	+6.7150 29679 20486 75481 43616	30
	9	−1.8652 86022 00135 20967 0656	30
	10	+3.9378 26046 44729 88708 2496	29
	11	−6.4437 15348 73194 36068 0448	28
	12	+8.2987 24312 76083 64633 088	27
	13	−8.5115 12115 65213 99623 68	26
	14	+7.0149 82512 90011 53536	25
	15	−4.6766 55008 60007 69024	24
	16	+2.5331 88129 65837 49888	23
	17	−1.1175 82998 37869 4848	22
	18	+4.0174 55222 92994 88	20
	19	−1.1746 94509 62864	19
	20	+2.7821 71207 0152	17
	21	−5.2993 73727 648	15
	22	+8.0293 54132 8	13
	23	−9.5209 7328	11
	24	+8.6240 7	9
	25	−5.7493 8	7
	26	+2.6535 6	5
	27	−7.56	2
	28	+1	0
29	1	+8.8417 61993 73970 19545 43616	30
	2	−1.2378 46679 12355 82736 36106 24	32
	3	+5.5703 10056 05601 22313 62478 08	32
	4	−1.2069 00512 14546 93167 95203 584	33
	5	+1.5086 25640 18183 66459 94004 48	33
	6	−1.2069 00512 14546 93167 95203 584	33
	7	+6.6092 17090 32042 72110 21352 96	32
	8	−2.5964 78142 62588 21186 15531 52	32
	9	+7.5730 61249 32548 95126 28633 6	31
	10	−1.6829 02499 85010 87805 84140 8	31
	11	+2.9068 31590 65018 78937 36243 2	30
	12	−3.9638 61259 97752 89460 03968	29
	13	+4.3195 92398 69346 10309 0176	28
	14	−3.7974 43866 98326 24447 488	27
	15	+2.7124 59904 98804 46033 92	26
	16	−1.5822 68277 90969 26853 12	25
	17	+7.5623 11622 36250 18048	23
	18	−2.9656 12400 92647 1296	22
	19	+9.5385 19418 18455 68	20

TABLE I. *Exact Values of the Coefficients,* $b_{n,l}$—Continued

n	l	$\beta_{n,l}$	$P_{n,l}$
29	20	−2.5101 36688 99593 6	19
	21	+5.3788 64333 56272	17
	22	−9.3140 50794 048	15
	23	+1.2885 05050 56	14
	24	−1.4005 48968	12
	25	+1.1671 2414	10
	26	−7.1823 024	7
	27	+3.0693 6	5
	28	−8.12	2
	29	+1	0
30	1	−2.6525 28598 12191 05863 63084 8	32
	2	+3.8461 66467 27677 03502 26472 96	33
	3	−1.7948 77684 72915 94967 72354 048	34
	4	+4.0384 74790 64060 88677 37796 608	34
	5	−5.2500 17227 83279 15280 59135 5904	34
	6	+4.3750 14356 52732 62733 82612 992	34
	7	−2.5000 08203 72990 07276 47207 424	34
	8	+1.0267 89083 67478 06559 97960 192	34
	9	−3.1374 11089 00627 42266 60433 92	33
	10	+7.3206 25874 34797 31955 41012 48	32
	11	−1.3310 22886 24508 60355 52911 36	32
	12	+1.9158 66275 65580 56572 35251 2	31
	13	−2.2106 14933 44900 65275 79136	30
	14	+2.0648 60102 67214 89543 3216	29
	15	−1.5732 26744 89306 58699 6736	28
	16	+9.8326 67155 58166 16872 96	26
	17	−5.0609 31624 19644 35155 2	25
	18	+2.1500 68990 67169 16896	24
	19	−7.5441 01721 65505 856	22
	20	+2.1838 18919 42646 432	21
	21	−5.1995 68855 77729 6	19
	22	+1.0129 03023 85272	18
	23	−1.6014 27705 696	16
	24	+2.0307 96003 6	14
	25	−2.0307 96003 6	12
	26	+1.5621 50772	10
	27	−8.9011 44	7
	28	+3.5322	5
	29	−8.7	2
	30	+1	0

Legend: $W_n(x) = \sum_{l=1}^{n} b_{n,l}\, x^l$

$b_{n,l} = \beta_{n,l} \times 10^{P_{n,l}}, \qquad 1 \leq \beta_{n,l} < 10$

TABLE II. *Roots of* W_n (ϵ/T)

n	l	$r_{n,l}$	C	$e_{n,l}$
2	1	2.00000 00000 0000	0	0
3	1	1.26794 91924 3112	0	5
3	2	4.73205 08075 6888	0	5
4	1	0.93582 22275 2409	0	5
4	2	3.30540 72893 3227	0	5
4	3	7.75877 04831 4363	0	5

TABLE II. *Roots of* W_n (ϵ/T)—Continued

n	l	$r_{n,l}$	C	$e_{n,l}$
5	1	0.74329 19279 8143	0	5
5	2	2.57163 50076 4627	0	5
5	3	5.73117 87516 8905	0	50
5	4	10.95389 43126 8321	1	6
6	1	0.61703 08532 7825	0	50
6	2	2.11296 59585 7838	0	50
6	3	4.61083 31510 1760	0	50
6	4	8.39906 69712 0486	0	50
6	5	14.26010 30659 2081	1	10
7	1	0.52766 81217 1117	0	50
7	2	1.79629 98096 4345	0	50
7	3	3.87664 15204 7699	0	50
7	4	6.91881 65667 0471	0	50
7	5	11.23461 04290 8311	1	6
7	6	17.64596 35523 8068	1	16
8	1	0.46102 42198 0496	0	50
8	2	1.56358 61896 5431	0	50
8	3	3.35205 05025 3674	0	50
8	4	5.91629 72490 2042	0	50
8	5	9.42069 93830 2156	0	50
8	6	14.19416 55480 0748	1	10
8	7	21.09217 69079 5447	1	22
9	1	0.40938 35732 0319	0	5
9	2	1.38496 31848 0312	0	5
9	3	2.95625 45561 6887	0	5
9	4	5.18194 31010 4007	0	5
9	5	8.16170 96881 4582	0	5
9	6	12.07005 51268 3715	1	7
9	7	17.24973 55261 4898	1	15
9	8	24.58595 52436 5281	1	30
10	1	0.36817 84529 4174	0	5
10	2	1.24335 79621 4047	0	5
10	3	2.64603 38413 8420	0	5
10	4	4.61688 25146 3485	0	50
10	5	7.22178 65393 9663	0	50
10	6	10.56732 08077 4184	1	56
10	7	14.83591 45152 6107	1	110
10	8	20.38218 19854 4899	1	207
10	9	28.11834 33810 4993	1	39
11	1	0.33452 86763 2476	0	5
11	2	1.12825 33558 7666	0	5
11	3	2.39586 99247 4731	0	5
11	4	4.16684 09879 2878	0	5
11	5	6.48735 30313 8081	0	5
11	6	9.42835 48133 3561	0	5
11	7	13.10172 35803 6780	1	9
11	8	17.69648 75668 4621	1	16
11	9	23.57778 70883 6019	1	28
11	10	31.68280 09748 3192	1	50

TABLE II. *Roots of* W_n (ϵ/T)—Continued

n	l	$r_{n,l}$	C	$e_{n,l}$
12	1	0.30652 67021 3005	0	5
12	2	1.03279 73987 7972	0	5
12	3	2.18961 19419 6840	0	5
12	4	3.79904 76060 5497	0	5
12	5	5.89491 11715 0275	0	5
12	6	8.52729 20080 7027	0	5
12	7	11.77100 65654 9068	1	7
12	8	15.74226 02870 2615	1	12
12	9	20.63580 56686 4129	1	213
12	10	26.82634 99493 7082	1	36
12	11	35.27439 07009 6510	1	62
13	1	0.28285 83482 3992	0	5
13	2	0.95232 60413 6462	0	5
13	3	2.01649 21385 7771	0	50
13	4	3.49235 40697 7799	0	50
13	5	5.40549 10200 1572	0	5
13	6	7.79281 39404 1242	0	5
13	7	10.70738 86889 8909	1	6
13	8	14.22715 23637 8996	1	10
13	9	18.47199 66342 2982	1	171
13	10	23.64178 37524 0122	1	279
13	11	30.12005 86261 0650	1	455
13	12	38.88928 43760 9550	1	76
14	1	0.26258 83981 7108	0	5
14	2	0.88355 03073 8774	0	5
14	3	1.86903 38151 9979	0	50
14	4	3.23241 86994 5120	0	50
14	5	4.99357 56070 7419	0	5
14	6	7.18061 04941 0079	0	5
14	7	9.83280 82510 2000	0	5
14	8	13.00562 24002 5865	1	85
14	9	16.77961 36411 7585	1	141
14	10	21.27791 17554 4831	1	226
14	11	26.70503 40989 2773	1	357
14	12	33.45278 49657 6721	1	559
14	13	42.52444 75660 1840	1	91
15	1	0.24503 30150 9310	0	5
15	2	0.82408 22200 4784	0	50
15	3	1.74187 24007 6449	0	50
15	4	3.00913 24590 1792	0	50
15	5	4.64163 40665 9064	0	50
15	6	6.66134 14966 2717	0	5
15	7	9.09830 27037 7053	0	5
15	8	11.99370 34331 8574	1	7
15	9	15.40498 80310 5145	1	119
15	10	19.41499 28281 6615	1	1890
15	11	24.14975 79520 2928	1	2920
15	12	29.81810 51196 7132	1	4460
15	13	36.81962 42569 9721	1	6760
15	14	46.17743 00169 8345	1	1060

TABLE II. *Roots of* W_n (ϵ/T)—Continued

n	l	$r_{n,l}$	C	$e_{n,l}$
16	1	0.22968 05054 2513	0	5
16	2	0.77214 49103 7539	0	5
16	3	1.63105 30990 6745	0	50
16	4	2.81514 45900 1225	0	50
16	5	4.33716 40773 3756	0	5
16	6	6.21464 27645 5924	0	5
16	7	8.47116 39813 4671	0	5
16	8	11.13833 19657 5080	1	6
16	9	14.25891 00216 2446	1	102
16	10	17.89205 34381 6953	1	1600
16	11	22.12262 01748 2668	1	2450
16	12	27.07931 14990 4742	1	3680
16	13	32.97497 35524 0162	1	5450
16	14	40.21658 37114 8568	1	8060
16	15	49.84622 17085 5745	1	1240
17	1	0.21614 03052 3945	0	5
17	2	0.72638 82432 5183	0	50
17	3	1.53359 31603 7353	0	50
17	4	2.64497 09986 1195	0	50
17	5	4.07097 81608 8018	0	50
17	6	5.82585 55151 0563	0	50
17	7	7.92850 41853 0666	0	5
17	8	10.40380 82899 5104	1	5
17	9	13.28466 10707 0697	1	88
17	10	16.61517 32168 6805	1	1380
17	11	20.45600 60200 3371	1	2090
17	12	24.89384 70253 3575	1	30900
17	13	30.05986 29201 6784	1	45000
17	14	36.17069 45436 3216	1	65600
17	15	43.64036 51841 5001	1	9540
17	16	53.52915 11602 5465	1	14300
18	1	0.20410 91085 7933	0	5
18	2	0.68576 75894 9453	0	5
18	3	1.44719 86793 8049	0	50
18	4	2.49443 08859 6248	0	50
18	5	3.83615 60319 9932	0	50
18	6	5.48411 54208 4047	0	5
18	7	7.45372 29495 8128	0	5
18	8	9.76498 43667 5180	0	5
18	9	12.44386 10519 8395	1	77
18	10	15.52434 30012 1457	1	1200
18	11	19.05171 87593 8568	1	1810
18	12	23.08800 07008 7174	1	26700
18	13	27.72155 63462 7824	1	38500
18	14	33.08585 98280 6407	1	54800
18	15	39.40115 37933 0861	1	77500
18	16	47.08820 96546 0141	1	1 11000
18	17	57.22481 18319 1492	1	16300

TABLE II. *Roots of* W_n (ϵ/T)—Continued

n	l	$r_{n,l}$	C	$e_{n,l}$
19	1	0.19334 77686 7901	0	5
19	2	0.64946 18107 9655	0	5
19	3	1.37007 54867 1706	0	5
19	4	2.36027 76257 7508	0	5
19	5	3.62737 58656 5560	0	5
19	6	5.18117 49562 2015	0	5
19	7	7.03443 06230 6566	0	5
19	8	9.20350 66842 8214	0	5
19	9	11.70932 47926 7879	1	69
19	10	14.57876 11306 9283	1	1060
19	11	17.84675 98274 1882	1	15900
19	12	21.55965 87534 0514	2	12
19	13	25.78070 10050 5240	2	17
19	14	30.59981 44865 1918	2	23
19	15	36.15265 02790 4671	2	33
19	16	42.66288 77797 8564	2	45
19	17	50.55778 33977 1778	2	64
19	18	60.93200 77265 0073	2	93
20	1	0.18366 51730 9616	0	50
20	2	0.61681 63821 2673	0	50
20	3	1.30079 94029 6677	0	500
20	4	2.23994 93990 0886	0	500
20	5	3.44047 19428 8150	0	500
20	6	4.91064 58796 7473	0	50
20	7	6.66116 06120 5821	0	50
20	8	8.70559 71771 2802	0	50
20	9	11.06110 86835 7575	1	612
20	10	13.74939 41200 8341	1	9450
20	11	16.79812 30059 5300	1	14100
20	12	20.24308 49633 0246	2	10
20	13	24.13156 72676 0267	2	15
20	14	28.52794 79713 1414	2	20
20	15	33.52361 76172 8890	2	28
20	16	39.25629 42907 8263	2	39
20	17	45.95295 06486 3834	2	53
20	18	54.04709 30247 0194	2	73
20	19	64.64971 24378 1605	2	105
21	1	0.17490 67523 8661	0	50
21	2	0.58730 30806 3812	0	50
21	3	1.23822 51018 3420	0	500
21	4	2.13139 62600 7712	0	500
21	5	3.27213 31335 1677	0	500
21	6	4.66749 44658 8837	0	500
21	7	6.32653 61976 7362	0	500
21	8	8.26067 09520 1371	0	50
21	9	10.48416 73812 0829	1	55
21	10	13.01484 87721 5828	1	8470
21	11	15.87508 70127 1499	1	1 26000
21	12	19.09325 19076 0624	1	18
21	13	22.70589 38881 7321	1	26
21	14	26.76117 02293 7952	1	357

TABLE II. *Roots of* W_n (ϵ/T)—Continued

n	l	$r_{n,l}$	C	$e_{n,l}$
21	15	31.32451 61370 0729	1	490
21	16	36.48870 33461 4824	1	666
21	17	42.39342 27457 7640	1	898
21	18	49.26881 38498 6849	1	1210
21	19	57.55442 09713 1555	1	1650
21	20	68.37703 78145 5231	1	235
22	1	0.16694 61390 1971	0	50
22	2	0.56049 08612 5406	0	50
22	3	1.18142 07583 5136	0	500
22	4	2.03295 74520 5904	0	500
22	5	3.11969 20402 1490	0	500
22	6	4.44769 50426 1758	0	500
22	7	6.02471 21681 1509	0	50
22	8	7.86043 77586 0933	0	50
22	9	9.96688 88709 4385	0	50
22	10	12.35891 86988 9685	1	765
22	11	15.05493 02175 9276	1	11300
22	12	18.07788 77154 4042	1	1 63000
22	13	21.45678 84662 6343	1	230
22	14	25.22887 63492 4893	1	318
22	15	29.44311 42999 0910	1	4310
22	16	34.16593 02974 6011	1	5830
22	17	39.49140 82536 2833	1	7810
22	18	45.56112 99787 8368	1	10400
22	19	52.60828 92435 3047	1	13900
22	20	61.07827 57571 7212	1	1860
22	21	72.11320 96312 7984	1	259
23	1	0.15967 90124 6359	0	50
23	2	0.53602 44507 2533	0	50
23	3	1.12962 03228 4296	0	500
23	4	1.94327 26194 9229	0	500
23	5	2.98097 42342 9980	0	500
23	6	4.24798 60975 3645	0	500
23	7	5.75098 94501 8811	0	500
23	8	7.49829 71042 4725	0	50
23	9	9.50013 87408 1857	0	5
23	10	11.76904 61305 3152	1	69
23	11	14.32037 80923 6802	1	1030
23	12	17.17304 70973 6204	1	14700
23	13	20.35054 65523 0563	2	10
23	14	23.88244 31429 7127	1	284
23	15	27.80661 96348 6417	1	3850
23	16	32.17279 15883 7417	1	5170
23	17	37.04832 56128 1149	1	68600
23	18	42.52855 84501 9839	1	90600
23	19	48.75689 07697 4619	1	1 19000
23	20	55.96946 93367 7959	1	15600
23	21	64.61735 31062 9634	1	20800
23	22	75.85754 84527 5272	1	2870

TABLE II. *Roots of* $W_n(\epsilon/T)$—Continued

n	l	$r_{n,l}$	C	$\epsilon_{n,l}$
24	1	0.15301 84895 9942	0	5
24	2	0.51360 83707 0681	0	5
24	3	1.08218 80942 3753	0	50
24	4	1.86121 64108 9897	0	50
24	5	2.85418 83431 3952	0	50
24	6	4.06569 47966 6047	0	50
24	7	5.50154 23910 2483	0	50
24	8	7.16891 89382 1722	0	5
24	9	9.07661 44660 1974	0	5
24	10	11.23531 54871 5978	1	63
24	11	13.65799 92756 6470	1	933
24	12	16.36046 86713 7067	1	13400
24	13	19.36209 01881 5387	1	1 88000
24	14	22.68683 57294 7187	1	2580
24	15	26.36479 43150 0290	1	3470
24	16	30.43444 26913 9403	1	46300
24	17	34.94620 46462 8518	1	61100
24	18	39.96833 92266 7395	1	80000
24	19	45.59738 43134 3094	1	1 04000
24	20	51.97849 53322 1339	1	1 36000
24	21	59.35068 02360 3490	1	1 77000
24	22	68.17050 51810 0240	1	23100
24	23	79.60945 44050 8424	1	3140
25	1	0.14689 16257 8146	0	5
25	2	0.49299 48461 2182	0	5
25	3	1.03859 20353 8078	0	50
25	4	1.78584 95467 3969	0	50
25	5	2.73784 45322 0507	0	50
25	6	3.89860 82901 4203	0	50
25	7	5.27322 15403 0644	0	50
25	8	6.86794 56382 7667	0	50
25	9	8.69039 61931 4098	0	5
25	10	10.74977 14241 0897	1	58
25	11	13.05715 36754 8216	1	852
25	12	15.62591 13581 6414	1	12200
25	13	18.47224 24144 4275	1	1 71000
25	14	21.61592 28735 8341	1	2340
25	15	25.08136 20017 9402	1	3140
25	16	28.89913 24031 5677	2	21
25	17	33.10826 73051 8389	2	27
25	18	37.75986 10435 6086	2	36
25	19	42.92302 51864 7370	2	46
25	20	48.69545 32211 6803	2	59
25	21	55.22399 77787 5012	2	76
25	22	62.75044 41754 7154	2	98
25	23	71.73671 59637 8361	2	129
25	24	83.36839 49267 2194	2	174
26	1	0.14123 67262 5809	0	5
26	2	0.47397 45378 8442	0	5
26	3	0.99838 34056 2151	0	50
26	4	1.71638 16871 9240	0	50

TABLE II. *Roots of* W_n (ϵ/T)—Continued

n	l	$r_{n,l}$	C	$e_{n,l}$
26	5	2.63069 31145 8470	0	50
26	6	3.74487 77262 0273	0	50
26	7	5.06340 83123 3855	0	50
26	8	6.59177 56068 7320	0	50
26	9	8.33662 63598 0514	0	5
26	10	10.30594 30256 1368	1	5
26	11	12.50927 80113 1609	1	78
26	12	14.95806 12826 7794	1	11200
26	13	17.66600 89930 4484	1	1 56000
26	14	20.64967 47456 6110	1	2140
26	15	23.92920 78044 9273	2	14
26	16	27.52942 09021 3584	2	19
26	17	31.48133 78942 1101	2	25
26	18	35.82451 67628 4751	2	32
26	19	40.61069 00156 5943	2	414
26	20	45.90978 68582 2976	2	526
26	21	51.82061 58754 0514	2	672
26	22	58.49167 48142 7643	2	854
26	23	66.16744 93598 1048	2	110
26	24	75.31508 13581 0590	2	141
26	25	87.13389 48199 8148	2	189
27	1	0.13600 12570 4444	0	5
27	2	0.45636 93581 3000	0	5
27	3	0.96118 10273 0921	0	50
27	4	1.65214 28929 8375	0	50
27	5	2.53167 76794 9454	0	50
27	6	3.60294 55780 8908	-0	50
27	7	4.86990 60826 2376	0	50
27	8	6.33740 31557 2642	0	50
27	9	8.01127 62543 8048	0	5
27	10	9.89850 34830 7608	0	5
27	11	12.00738 61381 8993	1	72
27	12	14.34778 78810 5506	1	10300
27	13	16.93144 76204 2762	1	143
27	14	19.77239 41164 6035	1	1950
27	15	22.88750 43962 7852	2	13
27	16	26.29727 10029 7123	2	17
27	17	30.02688 18438 7846	2	23
27	18	34.10778 47142 1721	2	292
27	19	38.58003 51906 4629	2	374
27	20	43.49597 58235 1797	2	472
27	21	48.92632 26588 5339	2	598
27	22	54.97096 30253 0608	2	758
27	23	61.77999 23077 3237	2	952
27	24	69.60052 54573 4751	2	1210
27	25	78.90479 29547 6498	2	156
27	26	90.90552 80995 1289	2	206
28	1	0.13114 02048 4340	0	5
28	2	0.44002 68410 2122	0	5
28	3	0.92665 89979 7975	0	5
28	4	1.59256 14353 9651	0	5

TABLE II. *Roots of* $W_n(\epsilon/T)$—Continued

n	l	$r_{n,l}$	C	$e_{n,l}$
28	5	2.43989 88598 8214	0	5
28	6	3.47148 98767 8727	0	5
28	7	4.69085 73686 2107	0	5
28	8	6.10229 80663 5926	0	5
28	9	7.71097 27209 1486	0	5
28	10	9.52302 14214 8264	0	5
28	11	11.54571 07983 2683	1	7
28	12	13.78762 26083 0963	1	10
28	13	16.25889 71483 2334	1	132
28	14	18.97155 08191 9327	1	1800
28	15	21.93989 61709 5973	2	12
28	16	25.18110 69743 2692	2	16
28	17	28.71599 40224 8103	2	21
28	18	32.57009 65076 4767	2	266
28	19	36.77526 28337 6414	2	338
28	20	41.37202 26709 0563	2	430
28	21	46.41330 39270 2954	2	5380
28	22	51.97058 20538 3640	2	6760
28	23	58.14479 02456 7904	2	8480
28	24	65.08757 79811 6239	2	10600
28	25	73.04862 35108 1621	2	1330
28	26	82.50512 46756 5939	2	171
28	27	94.68291 12582 9295	2	224
29	1	0.12661 47772 6314	0	5
29	2	0.42481 56894 2803	0	5
29	3	0.89453 69967 2982	0	5
29	4	1.53714 63597 6714	0	5
29	5	2.35458 60074 0362	0	5
29	6	3.34938 08943 4731	0	5
29	7	4.52468 00068 0930	0	5
29	8	5.88431 36574 5907	0	5
29	9	7.43286 73847 3363	0	5
29	10	9.17577 57820 5631	0	5
29	11	11.11944 12276 9518	1	6
29	12	13.27138 44455 9570	1	9
29	13	15.64043 65527 0260	1	123
29	14	18.23698 62111 4301	1	1670
29	15	21.07330 14271 4752	2	11
29	16	24.16395 46279 8314	2	15
29	17	27.52639 39971 5279	2	19
29	18	31.18172 74331 4674	2	244
29	19	35.15582 50255 9695	2	308
29	20	39.48091 56524 1023	2	3900
29	21	44.19798 25434 6970	2	4880
29	22	49.36051 71053 0558	2	6100
29	23	55.04072 95577 5849	2	7580
29	24	61.34056 90221 6674	2	9440
29	25	68.41319 89167 3268	2	11700
29	26	76.51079 93357 6897	2	14700
29	27	86.11542 16992 4401	2	1860
29	28	98.46569 76629 3931	2	242

TABLE II. *Roots of* $W_n(\epsilon/T)$—Continued

n	l	$r_{n,l}$	C	$e_{n,l}$
30	1	0.12239 13636 8232	0	5
30	2	0.41062 22202 4781	0	5
30	3	0.86457 25660 9784	0	5
30	4	1.48547 36509 1174	0	5
30	5	2.27507 48206 5174	0	5
30	6	3.23564 71256 6498	0	5
30	7	4.37001 73777 7174	0	5
30	8	5.68161 56855 1702	0	5
30	9	7.17453 65183 2250	0	5
30	10	8.85361 57459 8089	0	5
30	11	10.72452 73673 4938	1	6
30	12	12.79390 50612 0659	1	8
30	13	15.06949 56117 3280	1	11
30	14	17.56035 39915 4328	1	1540
30	15	20.27709 38755 2512	2	10
30	16	23.23221 33320 6006	2	14
30	17	26.44052 46096 4200	2	18
30	18	29.91973 14194 9155	2	224
30	19	33.69122 07222 7026	2	284
30	20	37.78117 59363 7412	2	3580
30	21	42.22218 88741 5065	2	4440
30	22	47.05567 82251 9296	2	55200
30	23	52.33567 93797 4686	2	68400
30	24	58.13511 50396 8783	2	84800
30	25	64.55692 28253 1261	2	1 04000
30	26	71.75574 28866 9527	2	12900
30	27	79.98619 96989 1509	2	16300
30	28	89.73509 12115 8820	2	2020
30	29	102.25357 28564 7799	2	260

Legend:

$r_{n,l}$: the lth root of the n-th order polynomial, W_n. The estimated error bound is

$$| r_{n,l} - \text{root} | \leq e_{n,l} \times 10^{-14}$$

The significance of the numbers in column C is:

0: the root was computed explicitly
1: the root was computed as $r_{n,l} = 10/r_{n,l}$
2: the root was computed as $r_{n,l} = 20/\bar{r}_{n,l}$

The formulae of equation (4) can be verified by induction based on the actual formal construction of successive derivatives of $\exp(-\epsilon/T)$ [1]. After these relations have been established they can be used with a simple induction argument to establish the correctness of explicit formulae which are convenient for other purposes, albeit not for calculations

$$b_{n,l} = \frac{(-1)^{n-l}(l+1)^2 \cdots (n-1)^2 n}{(n-l)!}. \tag{6}$$

The roots were located by the method of false position [2], which depends upon evaluation of the polynomials for successive approximations to the roots. The evaluations were computed by synthetic division using double precision arithmetic on the ORDVAC computer at the Aberdeen Proving Ground. The polynomials were calculated in one of three forms as described in the legend to Table II. This was necessary since the original form of the polynomial was relatively insensitive to changes in approximations for certain roots; that is, the remainder was too small

TABLE III. *Symmetric Function Check on Roots*

n	$b_{n,n-1}$	d_n
2	2	0
3	6	0
4	12	− 1
5	20	− 4
6	30	− 10
7	42	+ 11
8	56	− 6
9	72	+ 1
10	90	− 28
11	110	+ 5
12	132	+ 20
13	156	+ 47
14	182	+ 94
15	210	− 371
16	240	− 1263
17	272	−13059
18	306	+21289
19	342	+ 927
20	380	+ 27
21	420	− 6267
22	462	+39096
23	506	− 2411
24	552	−55267
25	600	− 5915
26	650	+23010
27	702	+ 1762
28	756	+ 223
29	812	− 4329
30	870	−18880

Legend: $b_{n,n-1} - \sum_{l=1}^{n} r_{n,l} = d_n \times 10^{-14}$

compared with the largest term in the synthetic division schema. When this occurred, one of the other forms of the polynomial was used in order to define the root more accurately.

The error estimates of Table II were based upon locating the root between two approximate values which gave remainders of opposite sign. Although this method of locating roots is inherently self-checking, a further numerical test was made by computing the symmetric function summations of Table III.

EDWIN S. CAMPBELL

New York University,
New York

E. M. FISCHBACH

Ballistic Research Laboratories,
Aberdeen, Maryland

J. O. HIRSCHFELDER

University of Wisconsin
Milwaukee, Wisconsin

1. J. O. HIRSCHFELDER & EDWIN S. CAMPBELL, *Analytical (power series) Solutions of Flame Propagation* (CM–784), University of Wisconsin Naval Research Laboratory, May 27, 1953, p. 17 ff.
2. F. B. HILDEBRAND, *Introduction to Numerical Analysis*, McGraw-Hill Book Co., New York, 1956, p. 446.

Bessel Functions for Large Arguments

By M. Goldstein and R. M. Thaler

Calculations of Bessel Functions of real order and argument for large values of the argument can be greatly facilitated by the use of the so called phase-amplitude method [1]. In this method two auxiliary functions, the amplitude and phase functions, are defined in terms of the regular and irregular solution of Bessel's equation. These auxiliary functions have the great advantage that they are monotonic functions of the argument; moreover, for large arguments these functions are slowly varying and, hence, easily amenable to computation and interpolation.

The Bessel Functions of the third kind, so-called Hankel Functions, are defined as follows:

$$H_\nu^{(1)}(x) = J_\nu(x) + iY_\nu(x) \tag{1.1}$$

and

$$H_\nu^{(2)}(x) = J_\nu(x) - iY_\nu(x). \tag{1.2}$$

These definitions suggest the alternative definition of the Bessel Function in terms of an amplitude and phase function, viz.,

$$A_\nu \equiv |H_\nu^{(1)}(x)| \equiv |H_\nu^{(2)}(x)| \tag{2.1}$$

or

$$H_\nu^{(1)} = A_\nu e^{+i\Phi_\nu} \tag{2.2}$$

$$H_\nu^{(2)} = A_\nu e^{-i\Phi_\nu} \tag{2.3}$$

so that

$$J_\nu(x) = A_\nu \cos \Phi_\nu \tag{2.4}$$

and

$$Y_\nu(x) = A_\nu \sin \Phi_\nu \tag{2.5}$$

The phase function $\Phi_\nu(x)$ obeys the first order differential equation

$$\frac{d\Phi_\nu}{dx} = \frac{2}{\pi x}[A_\nu(x)]^{-2} \tag{3}$$

Equation (3) represents the usual Wronskian condition. The amplitude function A_ν is seen to obey the following nonlinear second order differential equation

$$x^2\frac{d^2A_\nu(x)}{dx^2} + x\frac{dA_\nu(x)}{dx} + (x^2 - \nu^2)A_\nu(x) - \left(\frac{2}{\pi}\right)^2[A_\nu(x)]^{-3} = 0. \tag{4}$$

Equation (4) follows trivially from Bessel's differential equation and equation (3).

Received 4 June 1957. This work was performed under the auspices of the U. S. Atomic Energy Commission.

Defining $B_\nu(t)$ as

$$B_\nu(t) = \sqrt{\frac{\pi t}{2}}\, A_\nu(t), \tag{5}$$

where $t \equiv 1/x$, gives the following differential equations for $\Phi_\nu(t)$ and $B_\nu(t)$

$$\frac{d\Phi_\nu(t)}{dt} = -[B_\nu(t)]^{-2} \tag{6}$$

$$\frac{d^2B_\nu(t)}{dt^2} + \frac{1}{t^4}\left[1 - (\nu^2 - \tfrac{1}{4})t^2\right]B_\nu(t) - [B_\nu(t)]^{-3} = 0. \tag{7}$$

A boundary condition implied by equation (2.1) and the asymptotic behavior of $H_\nu(x)$ as $x \to \infty$ is

$$\lim_{t\to 0} \frac{d^2B_\nu(t)}{dt^2} = 0, \tag{8}$$

so that near $t = 0$, $B_\nu(t)$ is approximately given by the solution of equation (7) with

$$\frac{d^2B_\nu(t)}{dt^2} = 0.$$

This yields

$$B_\nu(t) \approx t\left[1 - (\nu^2 - \tfrac{1}{4})t^2\right]^{-1}. \tag{9}$$

Equation (9) represents the so-called JWKB approximation [2] to the amplitude function, and

$$\Phi_\nu(t) = \frac{1}{t} + \int_0^t \left(\frac{1}{u^2} - \frac{1}{B^2}\right)du - (\nu + \tfrac{1}{2})\frac{\pi}{2}. \tag{10}$$

The JWKB approximation for $\Phi_\nu(t)$ consists of the substitution of equation (9) in equation (10)

Power series expansion about $t = 0$ for equations (6, 7) are readily obtained. One finds that

$$\{B_\nu(t)/t\} = 1 + a_2t^2 + a_4t^4 + a_6t^6 + a_8t^8 + a_{10}t^{10} + \cdots \tag{11}$$

where

$$a_2 = \frac{1}{4}\alpha \tag{11.2}$$

$$a_4 = \frac{5}{32}\alpha^2 - \frac{3}{8}\alpha \tag{11.4}$$

$$a_6 = \frac{15}{128}\alpha^3 - \frac{37}{32}\alpha^2 + \frac{15}{8}\alpha \tag{11.6}$$

$$a_8 = \frac{195}{2048}\alpha^4 - \frac{611}{256}\alpha^3 + \frac{1821}{128}\alpha^2 - \frac{315}{16}\alpha \tag{11.8}$$

$$a_{10} = \frac{663}{8192}\alpha^5 - \frac{4199}{1024}\alpha^4 + \frac{29811}{512}\alpha^3 - \frac{2223}{8}\alpha^2 + \frac{2835}{8}\alpha \tag{11.10}$$

with

$$\alpha \equiv (\nu^2 - \tfrac{1}{4});$$

and

$$(12)\quad t\left[\Phi_\nu + (\nu + \tfrac{1}{2})\frac{\pi}{2}\right] = 1 + b_2 t^2 + b_4 t^4 + b_6 t^6 + b_8 t^8 + b_{10} t^{10} + \cdots$$

where

$$(12.2)\qquad b_2 = \frac{1}{2}\alpha$$

$$(12.4)\qquad b_4 = \frac{\alpha^2}{24} - \frac{\alpha}{4}$$

$$(12.6)\qquad b_6 = \frac{1}{80}\alpha^3 - \frac{7}{20}\alpha^2 + \frac{3}{4}\alpha$$

$$(12.8)\qquad b_8 = \frac{1}{1792}\alpha^4 - \frac{95}{224}\alpha^3 + \frac{807}{224}\alpha^2 - \frac{315}{56}\alpha$$

$$(12.10)\qquad b_{10} = \frac{7}{2304}\alpha^5 - \frac{35}{72}\alpha^4 + \frac{1975}{192}\alpha^3 - 58\,\alpha^2 + \frac{315}{4}\alpha.$$

Alternatively, the amplitude function may be expressed in terms of the following series

$$(13)\quad \frac{B_\nu(t)}{t} = [1 - \alpha t^2]^{-\frac{1}{4}}\{1 + c_2 t^2 + c_4 t^4 + c_6 t^6 + c_8 t^8 + c_{10} t^{10} + \cdots\},$$

where

$$(13.2)\qquad c_2 = 0$$

$$(13.4)\qquad c_4 = -\frac{3}{8}\alpha$$

$$(13.6)\qquad c_6 = -\frac{17}{16}\alpha^2 + \frac{15}{8}\alpha$$

$$(13.8)\qquad c_8 = -\frac{33}{16}\alpha^3 + \frac{1761}{128}\alpha^2 - \frac{315}{16}\alpha$$

$$(13.10)\qquad c_{10} = -\frac{27}{8}\alpha^4 + \frac{6975}{128}\alpha^3 - \frac{17469}{64}\alpha^2 + \frac{2835}{8}\alpha.$$

Equation (13) represents an expansion about the JWKB approximation. The series in Equation (13) is semi-convergent. Substitution of Equation (13) into Equation (6) may be used to obtain an explicit "JWKB" series for $\Phi_\nu(t)$. However, the analytic form of the coefficients is so complicated as to be of little practical value.

The amplitude and phase functions for $\nu = 0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1$ are plotted in figures 1 and 2 respectively. In figure 1 the function

$$\left[\frac{B_\nu(t)}{t}\right]$$

is plotted against t. In figure 2 the function

$$\left[t\left\{\Phi_\nu(t) + (\nu + \tfrac{1}{2})\,\frac{\pi}{2}\right\}\right]$$

is plotted against t. In terms of these functions the Bessel Functions $J_\nu(x)$ and $Y_\nu(x)$ become

$$J_\nu(x) = \left[\frac{B_\nu(t)}{t}\right]\sqrt{\frac{2t}{\pi}}\cos\Phi_\nu(t),$$

$$Y_\nu(x) = \left[\frac{B_\nu(t)}{t}\right]\sqrt{\frac{2t}{\pi}}\sin\Phi_\nu(t),$$

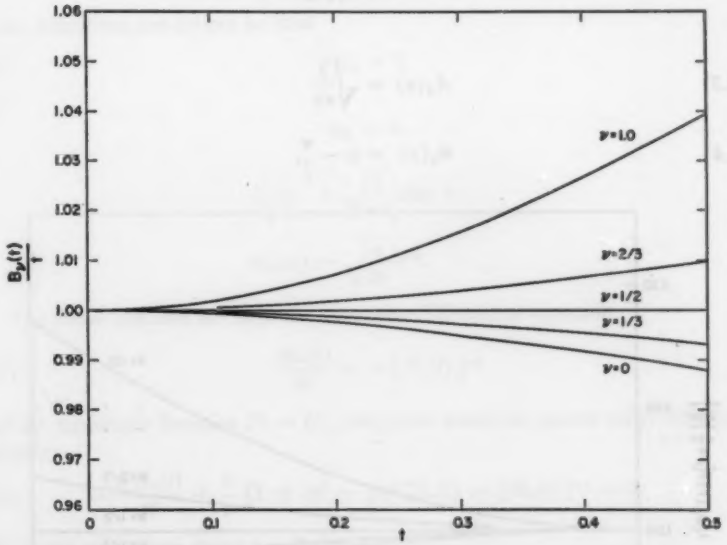

FIGURE 1. The amplitude function, $\frac{B_\nu(t)}{t}$ vs. t for several values of ν between zero and one.

The amplitude and phase functions are defined by the relations

$$J_\nu(x) = \left[\frac{B_\nu(t)}{t}\right]\sqrt{\frac{2t}{\pi}}\cos\Phi_\nu(t)$$

$$Y_\nu(x) = \left[\frac{B_\nu(t)}{t}\right]\sqrt{\frac{2t}{\pi}}\sin\Phi_\nu(t),$$

where

$$t = \frac{1}{x}.$$

with

$$t \equiv \frac{1}{x}.$$

For $|\nu| \leqq 1$, the series equation (11 or 13) and equation (12) yield eight figure accuracy in $J_\nu(1/t)$ and $Y_\nu(1/t)$ for $t \leqq \frac{1}{8}$, for the number of terms shown. This is in contrast with the usual asymptotic expansions which yield comparable accuracy only for $t \leqq 1/20$, and which converge much more slowly in the region of good convergence.

For $\nu = \frac{1}{2}$, $\alpha = 0$, so that the coefficients in equations (11) and (12) vanish. Therefore, for $\nu = \frac{1}{2}$ the series reduce to single terms, yielding

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x \tag{14.1}$$

$$Y_{\frac{1}{2}}(x) = -\sqrt{\frac{2}{\pi x}} \cos x \tag{14.2}$$

or

$$A_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \tag{14.3}$$

$$\Phi_{\frac{1}{2}}(x) = x - \frac{\pi}{2}. \tag{14.4}$$

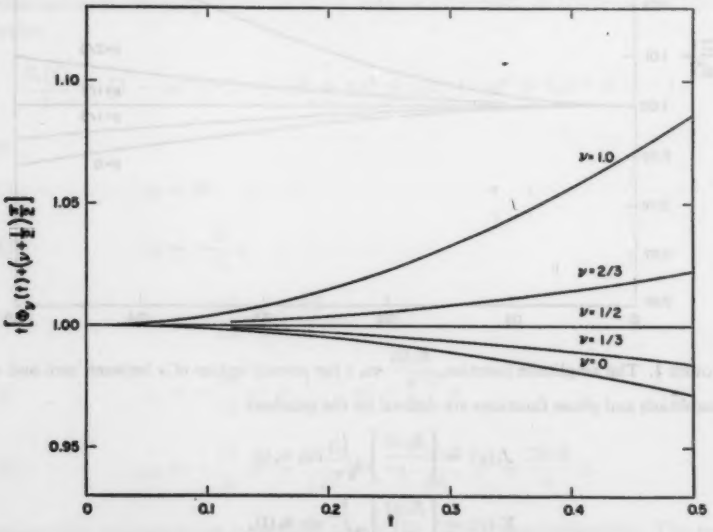

FIGURE 2. The phase function $t\left[\Phi_\nu(t) + (\nu + \frac{1}{2})\frac{\pi}{2}\right]$ vs. t, for several values of ν between zero and one.

It is also clear that the series are most rapidly convergent for ν close to $\frac{1}{2}$.

For computational purposes it is evident that greatest accuracy will result through the use of equations (11) and (12) for $|\nu| \leq 1$. The appropriate recurrence techniques may be used to obtain numerical values for $|\nu| > 1$.

Equations (11) and (12) may be regarded as starting series for the differential equations equations (6) and (7). For $|\nu| \leq 1$ equations (6) and (7) may be integrated numerically starting at $t \sim \frac{1}{8}$ and give good accuracy to $t \sim \frac{1}{2}$. Thus the method described above can be used to generate values of $J_\nu(x)$ and $Y_\nu(x)$ for $x \geq 2$.

The Bessel Functions of imaginary argument $I_\nu(x)$ and $K_\nu(x)$ are amenable to similar treatment. The phase-amplitude functions are defined as

$$I_\nu = \sqrt{\frac{2}{\pi x}}\, C_\nu \sinh \phi_\nu, \tag{15}$$

$$K_\nu = \sqrt{\frac{\pi}{2x}}\, C_\nu e^{-\phi_\nu}.$$

These definitions are chosen so that

$$C_{\frac{1}{2}} = 1 \tag{16}$$

and

$$\phi_{\frac{1}{2}} = x,$$

or

$$I_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sinh x$$

$$K_{\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2x}}\, e^{-x}.$$

The phase function ϕ_ν obeys the first order differential equation

$$\frac{d\phi_\nu(t)}{dt} = -[D_\nu(t)]^{-2} \tag{17}$$

and the amplitude function $D_\nu = tC_\nu$, obeys the nonlinear second order differential equation

$$-\frac{d^2D_\nu(t)}{dt^2} + \frac{1}{t^4}[1 + (\nu^2 - \tfrac{1}{4})t^2]D_\nu(t) - [D_\nu(t)]^{-3} = 0 \tag{18}$$

The series expansions about $t = 0$ are

$$\left\{\frac{D_\nu(t)}{t}\right\} = 1 - a_2t^2 + a_4t^4 - a_6t^6 + a_8t^8 - a_{10}t^{10} + \cdots, \tag{19}$$

and

$$t\phi_\nu(t) = 1 - b_2t^2 + b_4t^4 - b_6t^6 + b_8t^8 - b_{10}t^{10} + \cdots.$$

Alternatively, in the JWKB type approximation, the amplitude function is given by the series

$$\frac{D_\nu(t)}{t} = [1 + \alpha t^2]^{-\frac{1}{4}}\{1 - c_2t^2 + c_4t^4 - c_6t^6 + c_8t^8 - c_{10}t^{10} + \cdots\}, \tag{21}$$

where the a_n, b_n and c_n are defined as in equations (11), (12), and (13) and

$$\phi_\nu(t) = \frac{1}{t} + \int_0^t \left(\frac{1}{u^2} - \frac{1}{D^2}\right) du. \tag{22}$$

The amplitude and phase functions for $\nu = 0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1$ are plotted in figures 3 and 4 respectively. In figure 3 the function

$$[D_\nu(t)/t]$$

is plotted against t. In figure 4 the function

$$[t\phi_\nu(t)]$$

is plotted against t.

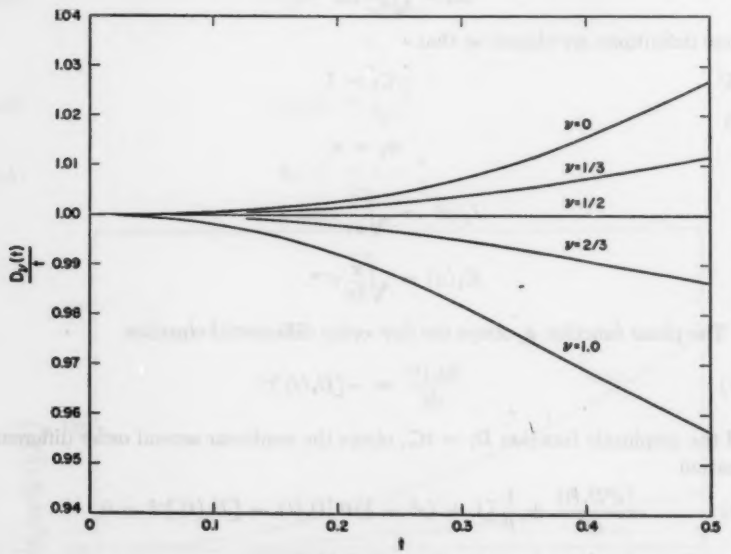

FIGURE 3. The amplitude function $\frac{D_\nu(t)}{t}$ vs. t for several values of ν between zero and one.

The amplitude and phase functions are defined by the relations

$$I_\nu(x) = \left[\frac{D_\nu(t)}{t}\right]\sqrt{\frac{2t}{\pi}}\sinh\phi_\nu(t)$$

$$K_\nu(x) = \left[\frac{D_\nu(t)}{t}\right]\sqrt{\frac{2t}{\pi}}\frac{\pi}{2}\exp\left[-\phi_\nu(t)\right]$$

where

$$t = \frac{1}{x}$$

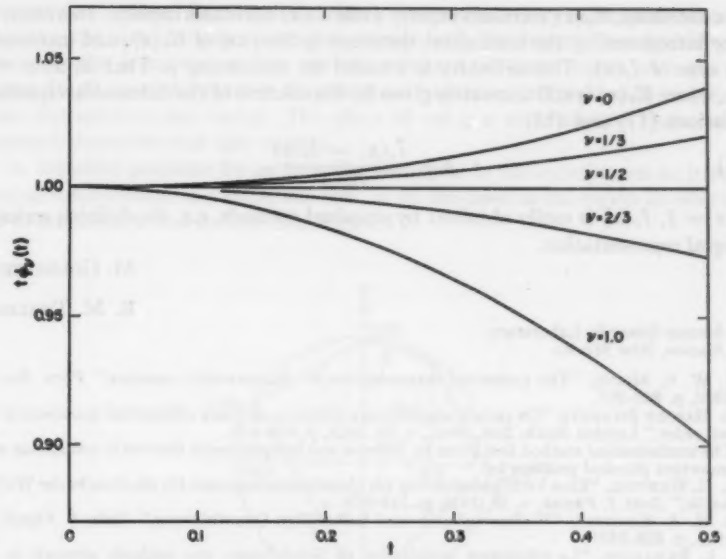

FIGURE 4. The phase function $t\phi_\nu(t)$ vs. t, for several values of ν between zero and one.

In terms of these functions the Bessel Functions of imaginary argument $I_\nu(x)$ and $K_\nu(x)$ become

$$I_\nu(x) = \left[\frac{D_\nu(t)}{t}\right]\sqrt{\frac{2t}{\pi}}\sinh\phi_\nu(t)$$

$$K_\nu(x) = \left[\frac{D_\nu(t)}{t}\right]\sqrt{\frac{2t}{\pi}}\frac{\pi}{2}e^{-\phi_\nu(t)}$$

with

$$t \equiv \frac{1}{x}.$$

The series, equations (19) and (20), can provide starting values for the numerical integration of the differential equations, equations (17) and (18). These series give seven figure accuracy for $|\nu| \leqq 1$ for $x \geqq 8$, for the number of terms shown. However, while straightforward numerical integration of equations (17) and (18) yields accurate values for $K_\nu(x)$ for $x \geqq 2$, the values for $I_\nu(x)$ rapidly deteriorate. The reason for this is clear. At $x = 8$, e.g., the starting values are accurate to eight figures. The functions $\bar{I}_\nu(x)$ and $\bar{K}_\nu(x)$ so calculated may then be written as

$$\bar{I}_\nu(x) = I_\nu(x) + \rho K_\nu(x)$$

$$\bar{K}_\nu(x) = K_\nu(x) + \sigma I_\nu(x),$$

where $\rho K_\nu(x)$ and $\sigma I_\nu(x)$ represent the truncation error, and ρ and σ are constants.

As x decreases, $K_\nu(x)$ increases rapidly while $I_\nu(x)$ decreases rapidly. Therefore, the error introduced by the truncation, decreases in the case of $K_\nu(x)$, and increases in the case of $I_\nu(x)$. This difficulty is avoided by calculating ρ. That is, at $x = 2$, e.g., where $K_\nu(x)$ is still accurately given by the solution of the differential equations, equations (17) and (18)

$$\rho = \frac{I_\nu(x) - I_\nu(x)}{K_\nu(x)}. \tag{24}$$

At $x = 2$, $I_\nu(x)$ is easily obtained by standard methods, e.g. the defining series, or integral representation.

M. GOLDSTEIN

R. M. THALER

Los Alamos Scientific Laboratory,
Los Alamos, New Mexico

1. W. E. MILNE, "The numerical determination of characteristic numbers," *Phys. Rev.*, v. 35, 1930, p. 863–867.
2. HAROLD JEFFREYS, "On certain approximate solutions of linear differential equations of the second order," London Math. Soc., *Proc.*, v. 23, 1923, p. 428–436.
The mathematical method first given by Jeffreys was independently derived in connection with an important physical problem by
a. G. WENTZEL, "Eine Verallgemeinerung der Quantenbedingungen für die Zwecke der Wellenmechanik," *Zeits. f. Physik*, v. 38, 1926, p. 518–529.
b. H. A. KRAMERS, "Wellenmechanik und halbzahlige Quantisierung," *Zeits. f. Physik*, v. 39, 1926, p. 828–840.
c. L. BRILLOUIN, "La mécanique ondulatoire de Schrödinger; une méthode générale de résolution par approximations sucessives," *Comptes Rendus*, v. 183, 1926, p. 24–26.
d. L. BRILLOUIN, "Remarques sur la mécanique ondulatoire," *J. de Physique*, v. 7, 1926, p. 353–368, and is commonly referred to as the JWKB approximation.

Numerical Experiments in Potential Theory Using the Nehari Estimates

By U. W. Hochstrasser

1. Introduction. It is well known that the Dirichlet problem can be solved by using a set of harmonic functions which are orthogonal over the boundary [7]. Prior to the availability of high speed computers, however, the method was not frequently used since generating sets of orthogonal functions by hand computations is very laborious. With the advent of high speed computers interest in the method has been renewed and several papers have appeared recently [1, 2, 3].

In using the method, the numerical analyst is particularly interested in estimates of the error committed by replacing infinite orthogonal expansions by finite ones. Nehari [1] recently has given such estimates for a number of Dirichlet problems. The purpose of the present paper is to report the results of some numerical experiments which were designed to investigate how close these estimates come to the actual errors in certain particular examples. An existing computer program [4] was used for the orthogonalization of a given set of vectors on the SEAC com-

Received 27 March 1957. The preparation of this paper was sponsored by the Office of Scientific Research of the Air Research and Development Command, United States Air Force, while the author was a guest worker at the National Bureau of Standards.

puter at the National Bureau of Standards. Several boundary value problems, each involving the same irregular pentagon, were solved. For two of these problems the exact solution is known so that one can compare the exact errors with the estimates. In all cases the number of orthogonal harmonic functions used to approximate the solution was varied. The effect of using a sophisticated choice of the harmonic functions was also studied.

2. Dirichlet problems for an irregular pentagon. In the experiments an irregular pentagon inscribable in a circle was used in all the cases as the region on which the Dirichlet problem was defined (see Figure 1).

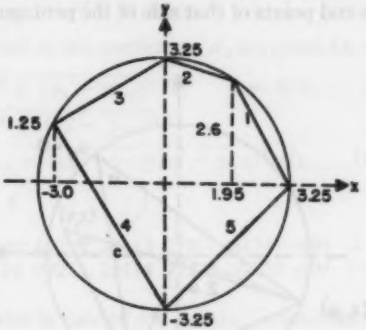

FIG. 1

For this region the following Dirichlet problems were considered:

(a) The boundary values are given by

$$U(x, y) = \text{Re } (\cos z) = \text{Re} \cos (x + iy) = \cos x \cosh y.$$

(b) The boundary values are given by

$$U(x, y) = \text{Re} \frac{1}{z + 4} = \frac{x + 4}{(x + 4)^2 + y^2}.$$

(c) The boundary values are equal to one on one side and zero on all the others. (The solutions of these problems are the harmonic measures for the sides.)

The solution of the first Dirichlet problem changes sign in the interior and on the boundary of the region. In the second case the solution is always positive on the whole region. The set of functions which were used to approximate the solution in the cases *a* and *b* consisted of harmonic polynomials obtained by orthonormalizing 1, Re z, Im z, Re z^2, Im z^2, $\cdots$ on the region. It was expected that with a few functions one would obtain a better approximation in the second case than in the first.

In order to find approximate values for the harmonic measures of the sides of the pentagon one can start from the harmonic measure of the circumscribed circle. Let the segment 2α be that part of the circumference of this circle which is cut out by the side of the pentagon with boundary values equal to 1. Then the harmonic measure u_0 which has the value 1 on the segment 2α and 0 on the rest of the circumference

is given by

$$u_0 = \frac{\varphi - \alpha}{\pi - 2\alpha}$$

(see Figure 2)

$$\varphi = \arccos\left[\frac{(x - x_2)(x - x_1) + (y - y_2)(y - y_1)}{\sqrt{[(x - x_1)^2 + (y - y_1)^2][(x - x_2)^2 + (y - y_2)^2]}}\right]$$

$$\alpha = \tfrac{1}{2}\arccos\left[\frac{x_1x_2 + y_1y_2}{x_1^2 + y_1^2}\right]$$

where φ is the angle belonging to any point in the region and (x_1, y_1), (x_2, y_2) are the coordinates of the end points of that side of the pentagon with boundary values

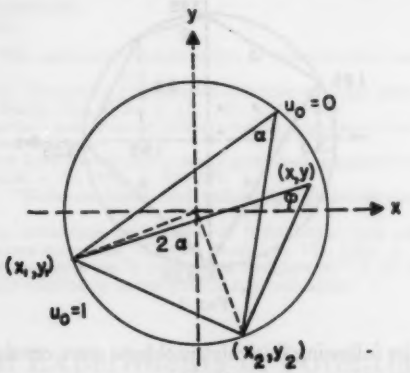

FIG. 2

equal to 1. In addition to these harmonic measures the harmonic polynomials 1, Re z, Im z, Re z^2, Im z^2, $\cdots$ were used for obtaining the set of approximating functions. These sets were orthonormalized with respect to the inner product

$$(f, g) = \int_c f(x, y)g(x, y)ds, \qquad ds^2 = dx^2 + dy^2$$

where the integration is along the boundary c of the region.

If n functions 1, Re z, Im z, Re(z^2), Im (z^2) $\cdots$ are used as basic set one obtains after orthonormalization a system of n orthonormal harmonic polynomials $u_\nu(x, y)$. The approximate solution of the Dirichlet problem is given then by the sum

$$u(x, y) = \sum_{\nu=1}^{n}\left[\int_c U(X, Y)u_\nu(X, Y)dS\right]u_\nu(x, y), \quad (dS^2 = dX^2 + dY^2).$$

For the numerical computations the contour integrals appearing in these computations must be replaced by an integration formula involving finite sums. Since the contour consists of straight lines, splitting the integral into five integrals corresponding to the five sides of the contour was suggested.

For their evaluation an eight point Gaussian-type integration rule was chosen involving the end points of the interval of integration as given by Radau [5].

This rule is exact for polynomials up to degree 13. It corresponds therefore to a seven point ordinary Gaussian integration rule. Since it involves the endpoints, the computational work for the eight point Radau rule and the seven point Gaussian rule is, however, the same for a closed contour. Radau's formula has the added convenience of using the corner points of the pentagon. Thus the contour integral is replaced by a sum over 35 products of the form

$$\int_c G(x, y)ds \sim \sum_{i=1}^{i=5}\sum_{j=1}^{j=7} w_{ij}G(A_j x_i + (1 - A_j)x_{i-1},\ A_j y_i + (1 - A_j)y_{i-1})$$

where (x_i, y_i) are the coordinates of the vertices of the pentagon

$$(x_0 = x_5, y_0 = y_5)$$

and the weights w_{ij} as well as the coefficients A_j are given by the integration rule

$$w_{ij} = \tfrac{1}{2}\sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}\,H_j \qquad j = 2, 3, \cdots, 7, i = 1, 2, \cdots, 5$$

$$w_{i1} = \tfrac{1}{2}\{\sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} + \sqrt{(x_{i-1} - x_{i-2})^2 + (y_{i-1} - y_{i-2})^2}\}H_1. \qquad (x_{-1} = x_4, y_{-1} = y_4)$$

j	1	2	3	4	5	6	7
H_j	.03571 428	.21070 422	.34112 270	.41245 880	$= H_4$	$= H_3$	$= H_2$
A_j	0	.06412 9925	.20414 991	.39535 039	$1 - A_4$	$1 - A_3$	$1 - A_2$

The orthonormalization is carried out by the Gram-Schmidt process: If f_i is a set of n vectors with N components then a system of orthonormal vectors u_i with components u_{ia}, is found by the formulas

$$u_1 = a_{11}f_1$$
$$u_2 = a_{21}f_1 + a_{22}f_2$$
$$\cdots$$
$$u_n = a_{n1}f_1 + a_{n2}f_2 + \cdots + a_{nn-1}f_{n-1} + a_{nn}f_n$$

where the a_{ij} are determined by the orthogonality and normalization condition

$$(u_i, u_j) = \delta_{ij} = 1; i = j \qquad i, j = 1, 2, \cdots, n.$$
$$= 0; i \neq j$$

Since in the continuous case the orthonormalization is with respect to the contour integral $\int_c f(x, y)u(x, y)ds$, the inner product is defined by

$$(f_i, u_j) = \sum_{k=1}^{N} w_k f_{ik}u_{jk} \qquad N = 35, w_1 = w_{1,1}w_2 = w_{1,2} \cdots w_7 = w_{1,7},$$
$$w_8 = w_{2,1} \quad \cdots \quad w_{15} = w_{2,7}$$
$$\cdots$$
$$w_{35} = w_{5,7}$$

where the weights w_k are given by the integration formula. Here the components of the vectors f_i are the values of the corresponding harmonic functions of the basic set 1, Re z, Im z, Re z^2 $\cdots$, in the cases a, b; or of the set u_0, Re z, $\cdots$, in case c, at the 35 points given by the integration rule.

The approximate solution $u(x, y)$ of the Dirichlet problem is given then by the sum

$$u(x, y) = \sum_{i=1}^{n} d_i f_i(x, y)$$

where

$$d_i = \sum_{k=1}^{i} (U, u_k) a_{ki} \qquad i = 1, 2, \cdots, n.$$

U is a vector the components of which are given by the boundary values of the Dirichlet problem in the points defined by the integration rule; $f_i(x, y)$ is the harmonic function corresponding to the vector f_i.

$u(x, y)$ has the property that among all the possible approximations of the form

$$\sum_{k=1}^{n} c_k f_k(x, y)$$

the deviation from the exact solution $U_e(x, y)$ is least in the least square sense:

$$\text{if } \Delta = U_e - \sum_{i=1}^{n} d_i f_i(x, y) = U_e - u$$

$$\text{then } (\Delta, \Delta) = (U_e - u, U_e - u) = \text{minimum.}$$

Nehari [1] has given the following estimate for the deviation in the region where the Dirichlet problem is defined

$$(U_e(x, y) - \sum_{i=1}^{n} d_i f_i(x, y))^2 \leqq (\Delta, \Delta) \cdot \left| \frac{1}{2\pi^2} \int_c \frac{dS}{(X - x)^2 + (Y - y)^2} - \sum_{i=1}^{n} u_i^2(x, y) \right|.$$

3. Numerical results. For each of the three cases a, b, and c the number n of harmonic functions in the basic set was varied in cases a and b from 3 to 11, in case c from 4 to 12. Due to the limitations in internal storage capacity it was not possible to go higher than 12. The functions were numbered in the following way

$$f_1 = 1, \quad f_2 = \text{Re}\, z = x, \quad f_3 = \text{Im}\, z = y, \quad f_4 = \text{Re}\, z^2 = x^2 - y^2, \quad f_5 = \text{Im}\, z^2 = 2xy$$

$$f_6 = \text{Re}\, z^3, \quad f_7 = \text{Im}\, z^3, \quad f_8 = \text{Re}\, z^4, \quad f_9 = \text{Im}\, z^4, \quad f_{10} = \text{Re}\, z^5, \quad f_{11} = \text{Im}\, z^5.$$

The variation of n was done in order to observe the convergence of the approximations to the exact solution of the Dirichlet problem. Due to the limited accuracy of the numerical computations (the orthonormalizing code works with floating decimal numbers containing only 8 digits) it must be expected that the improvement of the approximation with increasing n will level off until the addition of another

harmonic function f_i to the set will not improve the approximation at all. In the three cases this point has not yet been reached, as one can see from table 1, in which the norm of the vector consisting of the deviations in the points given by the integration rule is listed.

TABLE 1. *Norm of Error Vector* Δ
Harmonic Functions

Case	f_1, f_2, f_3	f_1 *to* f_6	f_1 *to* f_7	f_1 *to* f_9	f_1 *to* f_{11}
a	18.007	9.565	8.822	2.500	2.061
b	.351	.214	.157	.107	.076
c_1*	.176	.100	.094	.081	.057
c_2*	.126	.077	.026	.023	.022
c_3*	.219	.158	.149	.115	.113
c_4*		.099	.059	.038	.037
c_5*				.086	.078

* In these cases u_0 was used in addition to $f_1, \cdots, f_n$, and c_i is the case where the boundary values are one on the side i (see Figure 1) and zero on the rest of the boundary. One observes that in the case *a* the norm decreases more rapidly between $n = 7$ and 9. The norm for $n = 8$ is 2.517 and shows that the decrease occurs if $Re\ z^4$ is added to the set. The explanation for this behavior is probably that the function $Re\ z^4 = x^4 + y^4 - 6x^2y^2$ with its changes of sign along the lines $x = (\pm 1 \pm \sqrt{2})y$ (all combinations of signs are possible) fits particularly well the prescribed boundary conditions.

In table 2 the maximum error is given for the points on the boundary defined by the integration rule.

TABLE 2

Maximum Error in the Boundary Points Defined by the Integration Rule

Number of functions	*3*	*5*	*7*	*9*	*11*	*Max. bdry value*
Case						
a	11.832	5.739	4.825	1.480	.988	12.915
b	.232	.148	.109	−.087	−.066	.595
c_1*	−.146	−.065	−.078	−.068	−.052	1.0
c_2*	−.106	−.055	−.029	−.031	−.028	1.0
c_3*	−.154	−.115	−.095	−.091	−.095	1.0
c_4*		−.077	.032	.033	.034	1.0
c_5*				.070	.079	1.0

From this table one sees that with eleven vectors the maximum relative error in cases *a* and *b* is approximately 10% on the boundary. (Since the deviation has been evaluated only in a finite number of points it is of course possible that this error may be in some points somewhat larger.) In cases c_i the relative error is in most cases smaller than 10% due to the use of an additional harmonic function u_0 in the basic set. One remarks also that the maximum relative error for the first few approximations to the harmonic measures is much smaller than the error in the

cases a and b. This is due to the use of the corresponding harmonic measure u_0 for the circumscribed circle which gives a fairly good first approximation. From these observations one can conclude that from the numerical point of view the basic set of harmonic functions which has to be orthonormalized should not be chosen arbitrarily. The present computations show that a sophisticated choice of these functions creates considerably better approximations for the same number of orthonormal functions.

The Nehari estimates for the deviations in the point $x = y = 0$ are given for the cases a and b in table 3.

TABLE 3. *Deviations at $x = y = 0$ and Nehari Estimates.*

Harmonic functions	f_1 *to* f_3	f_1 *to* f_5	f_1 *to* f_7	f_1 *to* f_9	f_1 *to* f_{11}
Case					
a Nehari estimate	5.19	2.74	2.41	.643	.528
actual error (abs.)	.855	.495	.646	.0163	.0133
b Nehari estimate	.1010	.0612	.0428	.0276	.0195
actual error (abs.)	.0099	.0015	.0050	.0017	.0002

From this table one sees that the estimate is still considerably larger than the actual error and that it converges slower than the actual error with increasing number of harmonic functions. If one compares, however, this estimate to the maximum error on the boundary given by table 2 it is evident that the Nehari estimate gives better error bounds than the one which can be derived from the maximum modulus principle for harmonic functions. In table 4 the Nehari estimates for the cases a and b are given for a number of points in the interior of the pentagon and for 11 harmonic polynomials.

TABLE 4

Case ($y = 0$)	$x = -2$	$x = -1$	$x = 0$	$x = 1$	$x = 2$	$x = 3$
a Nehari estimate	1.798	.539	.528	.528	.614	1.631
actual error	.052	.094	.0133	.0105	.0181	.406
b Nehari estimate	.0663	.0199	.0195	.0195	.0226	.0601
actual error	.0091	.0023	.0002	.0021	.0038	.0086

Table 4 shows that the Nehari estimates become worse as the boundary of the region is approached if they are compared with the estimates given by the maximum modulus principle. With respect to the actual error, however, the Nehari estimates are further off near the middle of the region than near the boundary.

In table 5 the Nehari estimates for the harmonic measures of the sides in the case where 12 harmonic functions were used are given for $x = y = 0$ together with the maximum error in the points of the boundary given by the integration rule.

In these cases a comparison to the exact error is no longer possible since the exact solution is not known. The values show however that the Nehari estimate gives a better bound for the error at the origin than the one suggested by the maximum error on the boundary.

TABLE 5

Nehari Estimates at $x = y = 0$ and Maximum Error for Harmonic Measures

Case	C_1	C_2	C_3	C_4	C_5
Nehari estimate at $x = y = 0$	.0146	.00556	.0239	.00945	.0201
Maximum error	−.0520	−.0279	−.0945	.0335	.0788

4. Concluding Remarks. The best approximation in each of the seven cases gave values which are exact within at most ten per cent for many points of the region. Similar computations by P. Davis and P. Rabinowitz [6] show that with the given limited accuracy of the code the addition of further harmonic polynomials beyond Im z^5 would probably no longer improve this result considerably due to the influence of the round-off errors. In order to improve the accuracy one probably has to use at least double precision in the computations.

The numerical examples given on the previous pages show that the Nehari estimate gives a considerably better bound for the error in the interior of the region than the maximum modulus principle. With respect to the actual error the Nehari estimate is, however, still off by a factor which varied in the examples from 5 to almost 100. It may be that the situation improves if more harmonic polynomials are used with a double precision code, but this can be decided only by experiments using computing machines which have a larger high speed memory than the SEAC. Since the use of orthonormal functions for solving problems in potential theory seems particularly attractive if the solution has to be known only in certain points of the region it would be interesting to see how much the present results can be improved by using double precision codes on the faster large computers like the IBM 704 or Remington Rand 1103AF.

The author wishes to thank Dr. P. Davis for many valuable suggestions and G. Blakley, J. P. Menard, and B. Urban for performing the necessary computations.

U. W. HOCHSTRASSER

Computation Center,
University of Kansas,
Lawrence, Kansas

1. ZEEV NEHARI, *On the Numerical Solution of the Dirichlet Problem*, Proceedings of the Conference on Differential Equations, Univ. of Maryland Book Store, College Park, Md., 1956, p. 157–178.

2. S. BERGMAN, *The Kernel Function and Conformal Mapping*, Mathematical Surveys No. 5, Am. Math. Soc., New York, 1950.

3. L. E. PAYNE & H. F. WEINBERGER, "New bounds in harmonic and biharmonic problems," *Jn. Math. and Physics*, v. 33, 1955.

4. PHILIP DAVIS & PHILIP RABINOWITZ, "A multiple purpose orthonormalizing code and its uses," J. Assn. Comp. Machinery, v. 1, 1954, p. 183–191.

5. ZDENĚK KOPAL, *Numerical Analysis*, John Wiley and Sons, Inc., New York, 1955, p. 390–394.

6. P. DAVIS & P. RABINOWITZ, "Numerical experiments in potential theory using orthonormal functions," *Jn.* Wash. Acad. of Sciences, v. 46, 1956, p. 12–17.

7. S. ZAREMBA, "Sur le calcul numérique des fonctions demandées dans le problème de Dirichlet et le problème hydrodynamique," International de l'Académie des Science de Cracovie, *Bulletin*, 1909, p. 125–195.

Approximate Relations between Series and Integrals

By P. A. P. Moran

If we wish to evaluate numerically an integral of the form

$$I = \int_{-\infty}^{\infty} f(x)dx \tag{1}$$

we may replace it by the sum

$$S = \sum_{n=-\infty}^{\infty} hf(nh + \delta). \tag{2}$$

Here h is the interval width and δ a position constant. If $f(x)$ tends to zero sufficiently rapidly at $x = \pm\infty$ and is reasonably continuous we may expect (2) to give a good approximation to (1). In practice it is often found that (2) is a much closer approximation to (1) than would have been guessed simply by looking at the graph of $f(x)$. For example if

$$f(x) = (2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}x^2}$$

and $h = 2$ so that the 'panels' of the integration formula are very wide compared with the scale of variation of $f(x)$, we find that S, which is a periodic function of δ, never diverges from $I = 1$ by more than 0.015 whilst if $h = 1$, the divergence is never greater than 0.52×10^{-8} (Aitken [1], Yates [14]). On the other hand a function like $f(n) = \pi^{-1}(1 + n^2)^{-1}$ gives a much lower accuracy (0.37×10^{-2} for $h = 1$).

If $f(n)$ tends to zero like e^{-n^2}, say, all its derivatives will also tend to zero at $x = \pm\infty$, and use of the Euler-Maclaurin sum formula would suggest that $S = I$ exactly. The fallacy in the argument, explained by Goodwin [5], lies in the fact that the Euler-Maclaurin series is an asymptotic one only. Another reason for expecting a good approximation arises from the Gregory formula for numerical integration (Whittaker and Robinson [13], p. 143) which expresses the integral as a sum of ordinates together with correction terms involving the differences at each end.

Another approach to this situation is provided by Poisson's summation formula (Fettis [4] and Salzer [11]). This is

$$\int_0^{\infty} f(x)dx = h\left\{\tfrac{1}{2}f(o) + \sum_{k=1}^{\infty} f(kh)\right\} - 2\sum_{k=1}^{\infty} g(2k\pi h^{-1})$$

where

$$g(x) = \int_0^{\infty} f(t)\cos xt\,dt.$$

When $f(x)$ is even it will often happen that $g(x)$ is very small for large x and the approximation is then very good. Fettis, [4], gives a number of illustrations of the use of this fact.

Received November 15, 1956.

This phenomenon of finding such surprising accuracy is not confined to integrals of functions which behave like e^{-x^2}. In a problem concerning dilution series of viruses (Moran [9]) it became necessary to sum the series

$$\sum_{s=-\infty}^{\infty} (p^{3(2^{2s})} - p^{4(2^{2s})}) \tag{3}$$

where $0 < p < 1$. This series is in fact periodic in δ where $p = \exp -2^{\delta}$. Its mean value for varying δ can easily be found to be

$$\int_{-\infty}^{\infty} \{\exp [-3(2^x)] - \exp [-4(2^x)]\} dx = 2 - \log 3 \, (\log 2)^{-1} = 0.450375,$$

and an actual exploration of the field shows that the sum never diverges from this value by more than 0.000004.

Since any integral over a finite range can be turned into one over $(-\infty, \infty)$ this suggests a useful method of evaluating finite integrals. As an example consider the integral

$$J = \int_0^{\infty} \frac{\exp -u^2}{X + u} \, du$$

which is tabulated by Goodwin and Staton [6]. For the value $X = 0.2$ they give $J = 1.5948$. Putting $u = \exp x^3$ we get

$$J = 3 \int_{-\infty}^{\infty} \frac{x^2 \exp (-e^{3x^3} + x^3}{X + \exp x^3} \, dx.$$

Taking intervals of width 0.5 in x we get an estimate 1.58069. A better method is to put $u = e^x$ so that

$$J = \int_{-\infty}^{\infty} \frac{\exp (-e^{2u} + u)}{X + e^u} du$$

and again taking intervals of width 0.5 we get an estimate 1.594812 whose accuracy is at least 0.0001. One advantage of this method often is that when we want to evaluate an integral for a large number of values of some parameter in it (as X in the Goodwin-Staton integral) we can do most of the calculation of the ordinates once and for all. A second advantage is that the accuracy increases extremely rapidly with decreasing h. It is also clear that the problem of deciding what particular transformation into an integral on the range $(-\infty, \infty)$ is best, is one which would repay more study.

The difficulty in using this technique in practice is to obtain an estimate of the error. One method is to calculate S for $\delta = 0, 0.2, \cdots 0.8$, say (or 0.1, 0.3, $\cdots$, 0.9) and examine the variation by drawing a smooth curve. Since the mean value of S is equal to I, I must lie between the estimated largest and smallest value of the sum. The trouble with this is that it requires five to ten times as much work. However, it may be worth while to do this if the integral is required for many values of some parameter since it should be only necessary to estimate the error for two or three values of the parameter.

It may also, in some cases, be possible to obtain a still better estimate by the method invented by L. F. Richardson and called by him the 'deferred approach to

the limit' (for some interesting examples of this see Salzer [12] and Bolton and Scoins [2]). We calculate the sum S for a series of values of h ($h_0, 2h_0, \cdots, nh_0$ say) and use simple polynomial extrapolation to obtain an estimate for $h = 0$. Some care is necessary in this. The remainder is a function not only of h but also of the position of the grid. It could happen that an exact estimate was obtained for a particular value of h and that for smaller h the estimate moves away again from the true value. Moreover, the error may well be oscillatory. If, however, $f(n)$ is an even function and δ is put equal to zero it seems almost certain in most cases that the convergence will ultimately be monotonic and this method may be useful. It must also be remembered that in many cases the error will decrease faster than any power of h.

In some particular cases an estimate of the error can be calculated explicitly. This has been done in a valuable paper by Goodwin [5] who considers integrals of the form

$$\int_{-\infty}^{\infty} \exp(-x^2) f(x)\,dx$$

where $f(x)$ satisfies certain regularity conditions in the complex plane. In this case a good estimate for the error is, when $f(x)$ is taken as even, and $\delta = 0$,

$$E(h) = 2 \exp(-\pi^2 h^{-2}) \int_{-\infty}^{\infty} f(y - i\pi h^{-1}) \exp(-y^2)\,dy$$

and if $f(n)$ does not increase too rapidly as n tends to infinity this is approximately

$$E(h) \doteq 2\pi^{\frac{1}{2}} \exp(-\pi^2 h^{-2}) f(i\pi h^{-1}).$$

This result with $f(x) = 1$ can be used to evaluate more complicated integrals such as the integral of the multivariate normal distribution over the range

$$(x_1 > 0, \cdots, x_n > 0)$$

when the correlations have certain prescribed forms (Moran [8]), and also certain incomplete integrals of the bivariate and trivariate normal distribution with arbitrary lower bounds (Das [3]).

The trouble with these methods when applied to other functions is to evaluate the error. If we have an even function then it is clear from (2) that $\partial S/\partial\delta$ is zero when $\delta = 0$ and $\delta = \frac{1}{2}h$. Since S is a periodic function it will usually happen that one of these is a maximum and the other a minimum. In fact we can expand S in a Fourier cosine series and then it will usually be found that the first cosine term is much larger than all the rest. Thus we can be practically certain in most cases that if $f(n)$ is an even function its integral will lie between the limits

$$\sum_{n=-\infty}^{\infty} hf(nh) \quad \text{and} \quad \sum_{n=-\infty}^{\infty} hf((n + \tfrac{1}{2})h).$$

This can be proved to be true for the integral (Das [3]).

$$\frac{t \exp -\frac{1}{2}t^2}{2\pi} \int_{-\infty}^{\infty} \frac{\exp -\frac{1}{2}x^2}{t^2 + x^2}\,dx$$

and since this is equal to

$$(2\pi)^{-\frac{1}{2}} \int_t^\infty \exp -(\tfrac{1}{2}x^2)dx,$$

Das has found two very sharp inequalities for the probability integral. However, if we add these sums together we get a vastly more accurate result for the same amount of labour.

It is also of some interest to notice that if δ is taken as a random variable with a uniform distribution on the interval $(0, h)$ the mean value of S is the integral and its variance about this mean, like the error term above, can be found as a series in the Fourier transform of $f(x)$ (Moran [7]).

The above ideas can also be applied to integrals on the range $(0, 2\pi)$ of functions which are periodic with period 2π. In this case the sum over a finite number of points is often remarkably close to the integral. If, in addition $f(x)$ is an even function it is sometimes possible to obtain very sharp inequalities from above and below (Moran 10) for such functions as

$$I_0 = (2\pi)^{-1} \int_0^{2\pi} \exp (x \cos \theta) d\theta$$

and this technique may be useful in calculating such functions as

$$\int_0^t I_0(x)\, dx.$$

I am much indebted to a referee for some useful criticisms which have substantially improved this note.

P. A. P. MORAN

The Australian National University,
Canberra, A. C. T.

1. A. C. AITKEN, *Statistical Mathematics*, Oliver and Boyd, Edinburgh, 1939.
2. H. C. BOLTON & H. I. SCOINS, "Eigenvalues of differential equations by finite-difference methods," Camb. Phil. Soc., *Proc.*, v. 52, 1956, p. 215–229.
3. S. C. DAS, "The numerical evaluation of a class of integrals, II," Camb. Phil. Soc., *Proc.*, v. 52, 1956, p. 442–448.
4. H. E. FETTIS, "Numerical calculation of certain definite integrals by Poisson's summation formula," *MTAC*, v. 9, 1955, p. 85–91.
5. E. T. GOODWIN, "The evaluation of integrals of the form $\int_{-\infty}^{\infty} f(x)e^{-x^2}dx$," Camb. Phil. Soc., *Proc.*, v. 45, 1949, p. 241–245.
6. E. T. GOODWIN & J. STATON, "Table of $\int_0^\infty \frac{e^{-u^2}}{u+x}\, du$," *Quart. J. Mech. Appl. Math.*, v. 1 1948, p. 319–326.
7. P. A. P. MORAN, "Numerical integration by systematic sampling," Camb. Phil. Soc., *Proc.*, v. 46, 1950, p. 111–115.
8. P. A. P. MORAN, "The numerical evaluation of a class of integrals, I," Camb. Phil. Soc., *Proc.*, v. 52, 1956, p. 230–233.
9. P. A. P. MORAN, "The dilution assay of viruses," *J. Hygiene*, v. 52, 1954, p. 189–193.
10. P. A. P. MORAN, "Inequalities for Bessel functions," *Quart. J. Math.* (to appear).
11. H. E. SALZER, "Aerial pattern synthesis; use of Poisson's formula," *Wireless Engineer*, v 33, 1956, p. 240–244.
12. HERBERT E. SALZER, "A simple method for summing certain slowly convergent series," *J. Math. Phys.*, v. 33, 1954–55, p. 356–359.
13. E. T. WHITTAKER & G. ROBINSON, *The Calculus of Observations*, Blackie, London, 1932.
14. F. YATES, "Systematic sampling," Roy. Soc. Phil., *Trans.*, A, v. 241, 1948, p. 345–377.

A Search for Analogues of the Mathieu Groups

By E. T. Parker and Paul J. Nikolai

I. Introduction. This paper describes a search for simple groups of a certain type using the UNIVAC Scientific Computer, Model 1103A.

Except for alternating and symmetric groups, the Mathieu groups of degrees 11, 12, 23, and 24 are the only known finite, quadruply transitive groups. These four groups, together with the Mathieu group of degree 22, are the only known abstract finite simple groups belonging to no known infinite system. ([1] see "Mathieu groups" in index). The curious properties of these groups provide considerable motivation for a large-scale search by digital computer for analogous groups.

It can be noted that the degrees $p = 11$ and $p = 23$ are primes of the form $p = 2q + 1$, with q also a prime. The analogues to be considered are groups G with the following properties:

1) G is a transitive permutation group of degree $p = 2q + 1$, p and q primes;

2) G is a simple group, and not merely the group of order p;

3) G is a proper subgroup of the alternating group of degree p. Such groups have been studied by G. A. Miller [2] and first footnote thereof for earlier papers, and K. D. Fryer [3]. In these papers it is reported that no G exists for degrees 47, 59, and 83. Even for these fairly low degrees the computations are of such length that details were not published. For a fixed degree $p = 2q + 1$ the groups to be examined are relatively few. For each group tested, it was possible to construct a pair of generators in a form admirably suited to digital computation; this form is probably not in previous literature.

The program was run for all degrees $p = 2q + 1$, p and q primes, $23 \leq p \leq 1823$. The values of p are 23, 47, 59, 83, 107, 167, 179, 227, 263, 347, 359, 383, 467, 479, 503, 563, 587, 719, 839, 863, 887, 983, 1019, 1187, 1283, 1307, 1319, 1367, 1439, 1487, 1523, 1619, and 1823. *No G exists for any of these degrees*, with the sole exception of the Mathieu group of degree 23. Degree 23, for which the facts are well known, yielded an invaluable check on the validity of the program. Inclusion of the next three degrees provided a check both on the program and on the tedious computations of earlier investigators.

In Section II are the arguments which lead to description of a pair of generators of a group to be tested, and to the criteria by which such a group may be rejected as the alternating group. In Section III the code for the UNIVAC Scientific Computer is described, with comments on running the program. In Section IV the results are summarized.

II. Mathematical arguments. As noted in the Introduction, the groups G under consideration have properties:

1) G is a transitive permutation group of degree $p = 2q + 1 \geq 23$, p and q primes;

2) G is simple, and not of order p;

3) G is a proper subgroup of the alternating group of degree p.

Received 27 September 1957. The program was written and run during the summer of 1956, while the authors were employed by Remington Rand UNIVAC, Division of Sperry Rand Corporation, St. Paul, Minnesota. The authors wish to express their thanks to Remington Rand UNIVAC for providing time on the UNIVAC Scientific Computer, model 1103A.

The alternating group of degree p is disregarded as uninteresting in this search. Any group containing odd permutations has a normal subgroup of index 2, hence is composite if of order greater than 2. The group G is required to be simple (property 2), and of order $> p \geq 23$ (properties 1 and 2). The motivation for demanding property 3 is established.

The group G, being transitive of degree p (property 1), is of order divisible by p ([4] p. 170, Corollary). Thus G has a Sylow subgroup P belonging to the prime p. This subgroup P is of order p only, hence cyclic. Were P in the center of its normalizer in G, then the commutator subgroup K of G would be of order prime to p ([4] p. 327, Theorem II). A transitive group G of prime degree is primitive. The subgroup K is a normal subgroup of G, hence either the identity or transitive on the p letters ([4] p. 196, Theorem X). If K is the identity, then G is abelian and in turn of order p, which is ruled out by property 2. On the other hand, K transitive on the p letters implies that K is of order divisible by p, conflicting with the assumption that P is in the center of its normalizer. The only alternative is that G has an element b such that $b^{-1}Pb = P$, but $b^{-1}ab \neq a$, where a is some non-identical element of P.

It is expedient for computational purposes to identify the p letters permuted by G with the integers reduced modulo p. Without loss of generality a generator a of P may be chosen as $x \rightarrow x + 1 \pmod{p}$. Being in the holomorph of P, b is of the form $x \rightarrow tx \pmod{p}$, t a fixed integer $\not\equiv 0$ or $1 \pmod{p}$. Since $p - 1 = 2q$, q an odd prime, b is of order q, 2, or $2q$. The element b fixes only the zero residue class and permutes the remaining $2q$ letters in cycles of equal length. Hence the element b of even order is an odd permutation, excluded by property 3. Accordingly b is of order q. As the holomorph of P has a unique subgroup of index 2, b may be chosen as $x \rightarrow r^2x \pmod{p}$, r a primitive root of p. Thus chosen, b consists of two cycles, one permuting the quadratic residues of p, the other the quadratic non-residues.

The symmetric group of degree $2q + 1$, q an odd prime, has its Sylow q-subgroups of order q^2, generated by two disjoint cycles of length q. If G had as an element a single cycle of length q, then G would be $p - q + 1 = q + 2$ times transitive ([4] p. 207, Theorem I); noting that a transitive group of prime degree is primitive. In turn G would contain the alternating group [5]. Thus G has its Sylow q-subgroups of order q only; let Q denote the Sylow q-subgroup generated by b.

Assume that Q is in the center of its normalizer. Then K is of order prime to q. The normalizer in G of P is of order pq; hence the normalizer in K of P is of order only p. The second derived group of G is then of order prime to p, thus the identity alone. The group G must then be solvable ([4] p. 56), in conflict with property 2. Thus G has an element c such that $c^{-1}Qc = Q$, $c^{-1}bc \neq b$; then $c^{-1}bc = b^w$, $w \not\equiv 0, 1 \pmod{q}$. Two possibilities arise: either c fixes set-wise the two cycles of b, or c maps each cycle into the other. The search may with no real loss be restricted to minimal extensions of $\{a, b\}$, so that c may be assumed to be of prime order. (Should a G be found, it would then be of interest to drop the restriction that c be of prime order, repeating the computations with c' normalizing Q, c a power of c'. This was not done, as no G was found with c of prime order.) Any element c of prime order interchanging the two cycles of b must be of order 2, consisting of q transpositions;

such an element is not in the alternating group. Thus c may be chosen of prime order, fixing set-wise the cycles of quadratic residues and quadratic non-residues constituting b, and transforming the cycles into like powers of themselves. Accordingly c is of the form, $(x^2 \to rx^{2k}) \cdot (-x^2 \to nx^{2k}) \pmod p$, where exponent k is fixed, x ranges over all residues (mod p), r and n are a fixed quadratic residue and non-residue of p; -1 is a quadratic non-residue of p, since q is odd and in turn $p = 2q + 1 \equiv 3 \pmod 4$. The holomorph of the group of order q has a unique subgroup of order qh, h any divisor of $q - 1$; thus $\{b, c\} = \{b, c_1\}$, where c_1 is any power of any conjugate of c, provided that c and c_1 are of the same order. Thus it is necessary to consider only those c of the form $(x^2 \to rx^{2k}) \cdot (-x^2 \to -x^{2k}) \pmod p$. This form of the generator c was referred to in the Introduction. Other mathematically equivalent notations [1, 3] would be very awkward to simulate on a computer. A further observation reduces essentially to half the number of groups to be examined. Note that $y \to -y \pmod p$ transforms a into a^{-1} and $\{b, c\}$ into $\{b, c^*\}$, where c and c^* are respectively $(x^2 \to rx^{2k}) \cdot (-x^2 \to -x^{2k})$ and $(x^2 \to r^{-1}x^{2k}) \cdot (-x^2 \to -x^{2k}) \pmod p$. The powers of 4 include all quadratic residues of p. Accordingly, for a given p and exponent k, all groups to be examined have c of the form $(x^2 \to 4^i x^{2k}) \cdot (-x^2 \to -x^{2k}) \pmod p$, $0 \le i \le (p - 3)/4$. For each value of i, the permutation c belonging to $i + 1$ is obtainable by multiplying by 4 (mod p) the maps of all quadratic residues of p in the previous permutation.

For each prime divisor λ of $q - 1$ it is necessary to select only one value of k yielding c of order λ. Thus the number of groups to be examined for a given degree $p = 2q + 1$ is $(p + 1)/4$ times the number of prime divisors of $q - 1$.

For a given p and k, the c with $i = 0$ is of the simple form $y \to y^k \pmod p$, provided that k is odd, this condition being easily satisfied. As noted above, $p \equiv 3 \pmod 4$. Further, $q \equiv 1$ or $2 \pmod 3$, and $q \equiv 1 \pmod 3$ implies $p \equiv 0 \pmod 3$, which is impossible for p a prime ≥ 23. Hence $q \equiv 2 \pmod 3$ so that $p \equiv 2 \pmod 3$. Thus $p \equiv 11 \pmod{12}$. By quadratic reciprocity, -3 is a quadratic non-residue of p. The multiplicative group of residues prime to p being cyclic of order $2q$, -3 is a primitive root of p Thus c with $i = 0$ is $(0 \to 0) \cdot [(p - 3)^j \to ((p - 3)^k)^j] \pmod p$, $1 \le j \le p - 1$. This observation made unnecessary the direct generation in the computer of large blocks of k-th powers.

Any of the groups $\{a, b, c\}$ is generated by a and c alone. For $\{a, c\}$ is transitive on the p letters, the normalizer in G of P is of order pq, and P is not in the center of its normalizer in $\{a, c\}$.

Each group $\{a, c\}$ is simple ([3] Theorem 4.1), satisfying property 2. Thus any $\{a, c\}$ of the type considered is either the alternating group or a G having the three properties.

Since $p \equiv 2 \pmod 3$, $p - 2$ is composite. Thus any group $\{a, c\}$, with an element consisting of a single cycle of length a prime $\pi < p$, is the alternating group of degree p. The argument is given above for the special case $\pi = q$.

The program was designed to construct the permutation c for each group, then to decide whether any element $a^i c$ $(1 \le i \le p - 1)$ has any power consisting of a single cycle of odd prime length $\pi < p$. ($\pi = 2$ is ignored, since the alternating group has no element with only one cycle of even length.) Either a group was re-

jected as alternating; or, when testing all $a^i c$ failed to show the group alternating, an output of c was provided.

III. Description of the Code for the UNIVAC Scientific Computer. For each prime value $p = 2q + 1$, $23 \leq p \leq 1823$, the value $q - 1$ was factored. Then for each prime divisor λ of $q - 1$, an odd integer k was found such that $k^\lambda \equiv 1 \pmod{q}$, $k \not\equiv 1 \pmod{q}$. These 88 p, k pairs together with about 300 program instructions comprised the input.

The routines, with numerical designations, are described below for a given p, k pair.

1. *Generate and store the quadratic residues of p. Generate and store in addresses y, $1 \leq y \leq p - 1$, (high-speed magnetic core), the numbers $y^k \pmod{p}$; store p in address p.* Addresses 1 to p are thus loaded with c $(i = 0)$ for the first group of the block of $(p + 1)/4$ groups for the pair p, k. A copy of c is also stored on the drum for possible output and for proceeding to the next group (Routine 5). Routine 2 is now entered.
2. *Multiply the permutation in addresses 1 to p by the permutation* $a: x \to x + 1 \pmod{p}$. This routine is rapidly accomplished by using the Repeat instruction of UNIVAC Scientific to transmit the contents of address y to address $y - 1$, $2 \leq y \leq p$. The initial contents of address 1 are then placed at address p. Addresses 1 to p, having been loaded with $a^i c$, now contain $a^{i+1} c$, $0 \leq i \leq p - 2$. Proceed to Routine 3.
3. *Count and store the lengths of cycles of the permutation stored in addresses 1 to p.* The principal idea of this routine is the following: when an address has been accounted for in a cycle, the contents of this address are marked by a minus sign. Each cycle is traced through its course by proceeding from an address to the address indicated by the contents of the preceding. A sign test indicates the completion of a cycle. The next cycle is then begun at the first address whose constants are positive. The process necessarily terminates with the contents of addresses 1 to p all negative. Routine 4 is entered next.
4. *Test the set of cycle lengths to decide whether the group must be alternating.* This routine makes use of the criterion developed near the end of Section 2. Reformulated for computation, the proposition is: If $a^i c$ has a cycle of prime length $\pi < p$, and no other cycle of length divisible by π, then $\{a, c\}$ is the alternating group. $\pi = 2$ can be ignored, since the alternating group contains no element with only one cycle of even length.

A technique previously used by E. Lehmer ([6] p. 105) was employed to decide whether a given odd number is prime. Consecutive bits of the contents of consecutive addresses were loaded (as part of the program data) with one or zero according as consecutive odd numbers 1, 3, $\cdots$, 1823 are prime or composite. Having an odd number Z to be tested, $1/2\ (Z - 1)$ is divided by 36, the number of bits per word. The quotient then indicates the address and the remainder the bit in that address containing the information as to whether Z is prime.

Each cycle length L_i is checked as it is found by Routine 3. If L_i is a prime such that $(p - 1)/2 < L_i < p$, enter alternative B described below. In this case

all other cycle lengths are $<p$, hence indivisible by p. If $L_j = p$, enter alternative A; the group cannot be rejected on this permutation.

If Routine 3 is completed without the group having been rejected, the cycle lengths are summed. If this sum were not p, the machine would stop. This error stop never occurred. Now for any prime cycle length Z among the L_j, the product $\Pi_j L_j \not\equiv 0 \pmod{Z^2}$ indicates rejection of the group, and alternative B is entered. If the group is not rejected on any prime cycle length Z, enter Alternative A.

Alternative A: Return to Routine 2 except when all $a^i c$, $1 \leq i \leq p-1$, have been checked by $p-1$ passes through routines 2, 3, and 4. If all $a^i c$ have been tried, print a typewriter character indicating output, and punch a tape of the permutation c. Proceed to Routine 5.

Alternative B: Print a typewriter character indicating rejection of the group as alternating. Proceed to Routine 5.

5. *Generate c for the next group in the block of* $(p+1)/4$ *groups for this p, k pair.* Multiply the contents of each address y of c, $1 \leq y \leq p$ which is a quadratic residue of p by 4 (mod p). Write this new c over the old one; enter Routine 2 as before.

The program was run for 15,924 groups, requiring somewhat less than 6½ hours on the UNIVAC Scientific computer; that is, an average of about 1.5 seconds was consumed per group. For the groups of high degree, the typewriter indication of rejection of a group as alternating was nearly rhythmic. Some calculation of instruction times made it clear that most groups of high degree were rejected on ac, the first permutation tested.

IV. Results. It was demonstrated by the program that for $p = 2q + 1$, p and q primes, $23 \leq p \leq 1823$, the only G of the type sought is the Mathieu group of degree 23. An output tape was prepared for exactly one other group, namely $p = 47$, with c the initial permutation $x \to x^{45} \pmod{47}$. This occurred through no program or machine error. In this isolated case all $a^i c$ fail to satisfy the criterion for rejection. Longhand multiplication of a few permutations showed the group to be alternating. Actually it is a bit surprising that all but one group were rejected by this criterion.

The authors somewhat expected to construct at least one G with this program. It seemed plausible that the G might be quadruply transitive. Instead, the Mathieu groups remain isolated more than before. The results of the program yield good evidence for the conjecture: *An unsolvable transitive group of degree* $p = 2q + 1$, *p and q primes, is alternating or symmetric, except for* $p = 7$, 11, *and* 23. (There is exactly one exception for each of 7 and 23, and two for 11.)

ADDED IN PROOF. Paul J. Nikolai, currently with Wright Air Development Center, Wright-Patterson AFB, Ohio, has obtained computer time to continue the problem. The machine used is an 1103A with 12,288 words of core storage. The large storage capacity rendered unnecessary several time-consuming instructions of the earlier code. In addition, the authors improved the logical organization of the program. The result was a speed better than twice that reported above. As for degrees $23 < p \leq 1823$, all groups generated were rejected as alternating. The further degrees run are $p = 1907$, 2027, 2039, 2063, 2099, 2207, 2447, 2459,

2579, 2819, 2879, 2903, 2963, 2999, 3023, 3119, 3167, 3203, 3467, 3623, 3799, 3803, 3863, 3947, 4007, and 4079.

The Ohio State University,
Columbus, Ohio

1. ROBERT D. CARMICHAEL, *Introduction to the Theory of Groups of Finite Order*, Ginn and Co., New York, 1937.
2. G. A. MILLER, "Transitive groups of degree $p = 2q + 1$, p and q being prime numbers," *Quart. Jn. of Math.*, v. 39, 1908, p. 210–216.
3. K. D. FRYER, "A class of permutation groups of prime degree," *Canadian Jn. Math.*, v. 7, 1955, p. 24–34.
4. W. BURNSIDE, *Theory of Groups of Finite Order*, 2nd ed., Cambridge Univ. Press, Cambridge, 1911; Dover Publications, Inc., New York, 1955.
5. G. A. MILLER, "Limits of the degree of transitivity of substitution groups," Am. Math. Soc., *Bull.*, v. 22, 1915, p. 68–71.
6. EMMA LEHMER, "Number theory on the SWAC," *Proceedings of Symposia in Applied Mathematics*, v. 6, McGraw-Hill Book Co., Inc., New York, for the Am. Math. Soc., 1956, p. 103–108.

On the Accuracy of Implicit Difference Approximations to the Equation of Heat Flow

By Wolfgang Wasow

1. Introduction. Implicit finite difference methods are gaining favor in the numerical solution of initial value problems for partial differential equations, because their computational stability is comparatively insensitive to the mesh ratios. The difference equation problems of importance in the applications are mostly too complicated to permit a theoretical analysis of their approximating qualities.It is a reasonable working hypothesis that the properties of the simplest linear difference equations with constant coefficients are indicative of the situation to be expected in many more involved problems.

The present paper is devoted to a study of the convergence, the stability and the truncation error of implicit difference approximations to the initial value problem defined by the differential equation

$$L[u] \equiv u_t - u_{xx} = 0, \qquad 0 < x < \pi, \qquad 0 < t \leq T, \tag{1.1}$$

where the subscripts indicate partial differentiations, and the subsidiary conditions

$$u(x, 0 = f(x), \qquad u(0, t) = u(\pi, t) = 0, \tag{1.2}$$

in which $f(x)$ is a given function. The approximating difference equation to be considered is

$$\begin{aligned} L_{h,k}[U] \equiv {} & k^{-1}\{U(x, t + k) - U(x, t)\} - sk^{-2}\{U(x + h, t + k) \\ & - 2U(x, t + k) + U(x - h, t + k)\} - (1 - s)k^{-2}\{U(x + h, t) \\ & - 2U(x, t) + U(x - h, t)\} = 0, \qquad 0 < x < \pi, \qquad 0 < t \leq T. \end{aligned} \tag{1.3}$$

Received 1 May 1957. Sponsored by the United States Army under Contract No. DA-11-022-ORD-2059.

The function $U(x, t)$ must also satisfy the same subsidiary conditions as $u(x, t)$, i.e.,

$$U(x, 0) = f(x), \qquad U(0, t) = U(\pi, t) = 0. \tag{1.4}$$

It will be assumed throughout that π/h is an integer. Furthermore, h and k will be restricted to intervals $0 < h \leq h_1 < 1$, $0 < k \leq k_1 < 1$.

The cases $s = 0$, $s = 1$ have been extensively studied in the literature (See, e.g., [1], [2], [3]). A paper in process of publication by M. L. Juncosa and D. Young [4] contains an appraisal of the truncation error for $s = \frac{1}{2}$. This appraisal is less sharp than the one contained in theorem 2 of the present paper. P. Lax and R. Richtmyer [5] have applied their general theory of finite difference approximations for initial value problems to an equation slightly more general than (1.3). Their arguments are based on the norm $\left\{\int_0^\pi \phi^2(x, t)dx\right\}^{\frac{1}{2}}$. The elegant methods by means of which they study the convergence and stability can doubtless be extended to yield appraisals of the truncation error as well, but since uniform or pointwise appraisals are more satisfactory to the numerical analyst than bounds on the mean square norm, it is hoped that the results of the present paper are of some interest.

2. The Convergence of $U(x, t)$. If $f(x)$ is integrable and $\sum_{r=1}^{\infty} b_r \sin rx$ is its Fourier sine-series the function

$$u(x, t) = \sum_{r=1}^{\infty} \exp(-r^2 t) b_r \sin rx \tag{2.1}$$

is, for $t > 0$, a solution of the differential equation (1.1), as can be readily verified. Under rather mild additional conditions on $f(x)$ this function fulfills also the boundary conditions (1.2), at least in some weak sense [3]. Here, the stronger hypothesis

$$\sum_{r=1}^{\infty} |b_r| < \infty \tag{2.2}$$

will be made, which is, for instance, satisfied if

$$f(0) = f(\pi) = 0, \tag{2.3}$$

and $f(x)$ is continuous and has a piecewise continuous derivative.

The method of separation of variables leads to the formal trigonometric series

$$U(x, t) \sim \sum_{r=1}^{\infty} E_r^{t/k} b_r \sin rx \tag{2.4}$$

for the solution of (1.3) and (1.4). Here E_r is defined by

$$E_r = \frac{h^2 - 4(1 - s)k \sin^2(rh/2)}{h^2 + 4sk \sin^2(rh/2)}. \tag{2.5}$$

The derivation of (2.4) is straightforward and will be omitted.

If

$$2(1 - 2s)k - h^2 \leq 0, \tag{2.6}$$

and only then, it follows that

$$|E_r| \leq 1, \tag{2.7}$$

for all r. In this case the series in (2.4) is uniformly convergent, thanks to condition (2.2), and represents therefore the solution of the difference problem.

In order to prove the convergence of $U(x, t)$ to $u(x, t)$ it will now be shown that

$$\lim_{h,k\to 0} E_r^{t/k} = \exp(-r^2 t). \tag{2.8}$$

If k is so small that $|kr^2(1-s)| < 1$ and $|kr^2 s| < 1$, it follows from part a) of the Lemma in section 5 that

$$|\log E_r + 4k^{-2}k \sin^2(rh/2)| \leq c(4k^{-2}k \sin^2(rh/2))^2, \qquad c \text{ a constant.}$$

The right member is a bounded function of h, and it can therefore be concluded from this inequality that

$$\lim_{k\to 0} (t/k)(\log E_r + 4k^{-2}k \sin^2(rh/2)) = 0,$$

uniformly in h. Hence,

$$\lim_{k\to 0} E_r^{t/k} = \exp\{-4k^{-2}t \sin^2(rh/2)\}, \tag{2.9}$$

and (2.8) is now obtained by letting h tend to zero.

Since, under the hypotheses made, the series (2.4) converges uniformly, for all small h and k, the passage to the limit indicated by (2.8) can be performed termwise and leads to the series (2.1). Thus the following theorem has been proved.

THEOREM 1. *If the Fourier sine coefficients b_r of $f(x)$ satisfy the condition*

$$\sum_{r=1}^{\infty} |b_r| < \infty,$$

and if h and k tend to zero in such a way that $2(1-2s)k - h^2 \leq 0$ then

$$\lim_{h,k\to 0} U(x, t) = u(x, t),$$

where U and u solve the problems (1.3), (1.4), *and* (1.1), (1.2), *respectively.*

For $s < \frac{1}{2}$, the inequality (2.6) restricts the size of the ratio k/h^2. For $s \geq \frac{1}{2}$ theorem 1 proves the convergence of U to u for any form of the passage to the limit $h \to 0$, $k \to 0$. This goes beyond the frequently made statement that k/h^2 may have any constant value in this case. Observe that s is not restricted to the interval $0 \leq s \leq 1$.

COROLLARY. *Denote by $u_h(x, t)$ the solution of the difference-differential equation problem obtained from* (1.3) *by substituting U_t for $k^{-1}\{U(x, t+k) - U(x, t)\}$ and by imposing the subsidiary conditions* (1.4). *Then*

$$\lim_{h\to 0} U(x, t) = u_h(x, t),$$

provided (2.2) *is true.*

To prove this corollary it suffices to refer to (2.9) and to show that the uniformly convergent series obtained from (2.4) by substituting $\exp\{-4k^{-2}t \sin^2(rh/2)\}$ for

E_r satisfies the difference-differential equation and the boundary conditions (1.4). This can be done by simple verification. The condition (2.6) is, of course, true for all sufficiently small k, if h is kept fixed.

It is easily seen that an analogous corollary could be stated concerning the difference-differential equation obtained if the difference quotients with respect to x in (1.3) are replaced by the corresponding derivatives. Here, however, the condition $s \geq \frac{1}{2}$ has to be imposed in order to maintain the inquality (2.6). The solution of this problem is the series obtained from (2.4) by substituting

$$\frac{1 - (1 - s)kr^2}{1 + skr^2}$$

for E_r.

Such difference-differential equations are sometimes used on analog machines. The arguments of this section justify these procedures mathematically.

3. The Truncation Error. In this section the order of magnitude of the truncation error $U - u$ will be determined (see theorem 2 below). The condition (2.6) is known to be necessary at least for $s = 0$, if convergence for all reasonably regular initial functions $f(x)$ is desired. Very likely it is necessary for all $0 < s < \frac{1}{2}$. Various heuristic arguments lead to the conjecture that $s = \frac{1}{2}$ is the most favorable choice as far as the order of magnitude of the truncation error is concerned. For instance, if $L_{h,k}[u]$ in (1.3) is developed in powers of h and k, where u is a solution of (1.1), the resulting series begins with terms of order $0(k + h^2)$, except if $s = \frac{1}{2}$, in which case the leading terms are $0(k^2 + h^2)$. The results of this section will confirm the expected exceptional role of the value $s = \frac{1}{2}$. Since the purpose of implicit methods is to gain more freedom in the choice of h and k, the range $0 < s < \frac{1}{2}$, in which the condition (2.6) is necessary, is of little interest. Throughout this section the inequalities

$$\frac{1}{2} \leq s \leq 1 \tag{3.1}$$

will therefore be imposed.

In view of theorem 1 the convergence of U to u takes place no matter how h and k tend to zero, if (3.1) is true. However, the order of magnitude of $U - u$ is very likely affected if h is esponentially small with respect to k. Here the inequality

$$h/k \geq p > 0, \qquad p \text{ a constant} \tag{3.2}$$

will be assumed to be true. Without too much complication the arguments apply also to the more general case that $h^\beta/k \geq p > 0, \beta > 0$. The restriction to (3.2) is motivated by the desire to avoid involved formulas. The condition (3.2) is considerably weaker than the hypothesis $k/h^2 = const$ frequently met in this connection. It is also milder than the assumption $k = 0(h/\log h)$ made in [4].

Finally, the conditions on $f(x)$ must be tightened in this section. It will be assumed that

$$| b_r | \leq Kr^{-3}. \tag{3.3}$$

This is, for instance, the case if (2.3) is true and $f(x)$ possesses an integrable third

derivative in $0 \leq x \leq \pi$, for three integrations by parts show that then

$$b_r = \frac{2}{\pi}\int_0^\pi f(x) \sin rx dx = \frac{2}{\pi} r^{-3}[(-1)^r f''(\pi) - f''(0)] - \frac{2}{\pi} r^{-3}\int_0^\pi f'''(x) \cos rx dx. \tag{3.4}$$

The following theorem will now be proved.

THEOREM 2. *Let $\frac{1}{2} \leq s \leq 1$, $h/k \geq p > 0$, $k \leq t \leq T$, $h \leq h_1 < p/2$. Assume that $f(x)$ possesses an absolutely integrable bounded third derivative and that $f(0) = f(\pi) = 0$. Then*

$$| U(x, t) - u(x, t) | \leq ct^{-1}(h^2 + k^\alpha),$$

where $\alpha = 1$ for $s \neq \frac{1}{2}$, and $\alpha = 2$ for $s = \frac{1}{2}$. Also,

$$c = M \sup_{0 \leq x \leq \pi} (|f''(x)| + |f'''(x)|).$$

The constant M depends on s, p and T only.

The truncation error will be appraised by splitting the series in

$$U(x, t) - u(x, t) = \sum_{r=1}^{\infty} (E_r^{t/k} - \exp(-r^2 t))b_r \sin rx \tag{3.5}$$

into four parts V_1, V_2, V_3, V_4 corresponding, respectively, to the ranges $r \leq r_1$, $r_1 < r \leq r_2$, $r_2 < r \leq r_3$, $r_3 < r$. The integers r_j, $j = 1, \cdots, 4$, are defined below.

a) *The range $r \leq r_1$.* The numerator

$$h^2 - 4(1 - s)k \sin^2 (rh/2)$$

of E_r vanishes if

$$r = 2/h \arcsin \frac{h}{2\sqrt{(1 - s)k}}. \tag{3.6}$$

Let r_0 be the smallest positive determination of the right member of (3.6) if the latter is real, i.e., if $h/(2\sqrt{(1 - s)k}) \leq 1$. Otherwise, set $r_0 = \infty$. If r_0 is finite it lies in the interval

$$\frac{1}{\sqrt{(1 - s)k}} \leq r_0 \leq \pi/2 \frac{1}{\sqrt{(1 - s)k}}, \tag{3.7}$$

because $1 \leq (\arcsin q)/q \leq \pi/2$, for $0 \leq q \leq 1$, if the principal value of $\arcsin q$ is taken. Define r_1 by

$$r_1 = [\sqrt{\theta p/h}], \qquad 0 < \theta < 1. \tag{3.8}$$

Here $[z]$ designates the largest integer not exceeding z.

The letters c_j, $j = 1, 2, \cdots$, below denote positive constants that may depend on s, p, and T, but not on h, k, x, t, or $f(x)$.

In order to appraise V_1 let $r \leq r_1$, and observe that, by (3.2), (3.7) and (3.8),

$$r_1 \leq r_0/\sqrt{2}. \tag{3.9}$$

Also, if z_τ is defined by

$$z_\tau = 4kh^{-2}\sin^2(\tau h/2), \tag{3.10}$$

it follows from (3.8) that

$$z_\tau \leq kr_1^2 \leq \theta, \qquad \text{for } \tau \leq \tau_1. \tag{3.11}$$

Since

$$E_\tau = \frac{1-(1-s)z_\tau}{1+sz_\tau},$$

an application of the Lemma of section 5 shows that

$$E_\tau^{t/k} = \exp\left[-t(z_\tau - R)/k\right], \tag{3.12}$$

where R satisfies the inequalities

$$-c_1 z_\tau^3 \leq R \leq \tfrac{1}{2}s^2 z_\tau^2, \qquad \tau \leq \tau_1. \tag{3.13}$$

If S is defined by

$$S = \tau^2 - z_\tau k^{-1} + Rk^{-1}, \tag{3.14}$$

formula (3.12) can be written

$$E_\tau^{t/k} = \exp(-\tau^2 t + St), \tag{3.15}$$

so that

$$E_\tau^{t/k} - \exp(-\tau^2 t) = \exp(-\tau^2 t)(\exp St - 1) = St\exp((S^* - \tau^2)t), \tag{3.16}$$

with S^* having a value between 0 and S. The quantity S can be appraised by inserting for z_τ its expression from (3.10) and by applying the inequalities (3.13) and

$$0 \leq \tau^2 - \sin^2\tau \leq \frac{1}{6}\tau^4.$$

This leads to

$$-c_1 k\tau^4 \leq S \leq \frac{1}{24}\tau^4 h^2 + \tfrac{1}{2}s^2 k\tau^4 \tag{3.17}$$

and therefore, if (3.2), (3.8) and (3.11) are recalled, to

$$S^* \leq S \leq \frac{1}{24}r_1^4 h^2 + \tfrac{1}{2}s^2 k r_1^2 \tau^2 \leq \frac{1}{24}p^2 + \tfrac{1}{2}s^2\tau^2. \tag{3.18}$$

Applying the inequalities (3.17) and (3.18) to (3.16) the appraisal

$$|E_\tau^{t/k} - \exp(-\tau^2 t)| \leq tc_3(h^2 + k)\tau^4 \exp(-3\tau^2 t/4), \qquad \tau \leq \tau_1 \tag{3.19}$$

is obtained.

The expected improvement in the order of magnitude, when $s = \frac{1}{2}$ can be based on part c) of the Lemma in section 5, according to which

$$-c_2 z_\tau^3 \leq R < 0, \qquad s = \tfrac{1}{2}$$

so that (3.17) can be replaced by

$$-c_2k^2r^4 \leq S \leq \frac{1}{24}r^4h^2 \leq \frac{1}{24}p^2, \qquad s = \tfrac{1}{2}$$

Inserted into (3.16) this leads to

$$(3.20) \qquad |E_r^{t/k} - \exp(-r^2t)| \leq tc_4(r^4h^2 + r^6k^2)\exp(-r^2t), \quad s = \tfrac{1}{2}, \quad r \leq r_1.$$

The quantity V_1 is now readily appraised by means of (3.3), (3.19), and (3.20). One has

$$(3.21) \qquad |V_1| = \left|\sum_{r=1}^{r_1}(E_r^{t/k} - \exp(-r^2t))b_r \sin rx\right|$$

$$\leq tc_3(h^2 + k)K\sum_{r=1}^{r_1} r\exp -3r^2t/4$$

$$\leq tc_3(h^2 + k)K\int_0^\infty r\exp(-3r^2t/4)dr = c_3K(h^2 + k),$$

and for $s = \frac{1}{2}$, in addition, in view of (3.20),

$$(3.22) \qquad |V_1| \leq tc_5K\left\{h^2\int_0^\infty r\exp(-r^2t)dr + k^2\int_0^\infty r^3\exp(-r^2t)dr\right\}$$

$$\leq t^{-1}c_6K(h^2 + k^2), \qquad s = \tfrac{1}{2}.$$

b) *The range* $r_1 < r \leq r_2$. The quantity E_r is a decreasing function of r and positive for $r < r_0$. Hence, $E_r^{t/k}$ is a positive decreasing function of r in $r < r_0$. Let r_2 be defined by

$$(3.23) \qquad r_2 = [r_0]$$

(so that $r_2 = \infty$, if $r_0 = \infty$), then

$$0 \leq E_r^{t/k} \leq E_{r_1}^{t/k}, \qquad r_1 \leq r \leq r_2$$

or, in view of (3.19),

$$|E_r^{t/k} - \exp(-r^2t)| \leq E_{r_1}^{t/k} + \exp(-r_1^2t) \leq |E_{r_1}^{t/k} - \exp(-r_1^2t)| + \exp(-r_1^2t)$$

$$\leq \{tc_3(h^2 + k)r_1^4 + 1\}\exp(-r_1^2t/2).$$

By virtue of formulas (3.3) and (3.5) this implies

$$|V_2| \leq \{tc_3(h^2 + k)r_1^4 + 1\}\exp(-r_1^2t/2)K\sum_{r=r_1}^{r_2} r^{-3}.$$

The last summation is less than $\int_{r_1}^\infty r^{-3}dr = 2r_1^{-2}$. Now we assume that $h_1 \leq \theta p/16$, so that $r_1^2 > \theta p/h - 2\sqrt{\theta p/h} + 1 \geq \theta p/2h$. Then the inequality for $|V_2|$ becomes

$$(3.24) \qquad |V_2| \leq \{c_7(h^2 + k) + (8h/(\theta p))\exp(-t\theta p/4h)\}K$$

$$\leq c_8t^{-1}K(h^2 + k), \qquad \text{for } r_1 \leq r \leq r_2.$$

c) *The range* $r_2 < r \leq r_3$. The set $r > r_2$ is empty, if $r_0 = \infty$. In the opposite case, E_r is a decreasing negative function in $r_0 < r < \pi/h$. Let

$$r_3 = \pi/h$$

Then, for $r_2 < r \leq r_3$,

$$|E_r^{t/k}| = |E_r|^{t/k} = \left(\frac{4(1-s)k\sin^2(rh/2) - h^2}{4sk\sin^2(rh/2) + h^2}\right)^{t/k}$$

$$\leq \left(\frac{r^2k(1-s)-1}{r^2ks+1}\right)^{t/k} \leq \left(\frac{r^2ks-1}{r^2ks+1}\right)^{t/k}.$$

Here we have made use of the monotonic dependence of E_r on the variable

$$4k\sin 2rh/2.$$

As $r^2ks > r_0ks > s/(1-s)$, part b) of the Lemma in section 5 can be applied to the last bound for $|E_r^{t/k}|$, provided $s > \frac{1}{2}$. This leads to

$$|E_r^{t/k}| \leq \left(\frac{1-1/r^2ks}{1+1/r^2ks}\right)^{t/k} \leq \exp(-2t/r^2k^2s), \qquad r_2 < r \leq r_3. \tag{3.25}$$

By continuity, formula (3.25) remains true for $s = \frac{1}{2}$. Hence, using (3.3) and (3.25)

$$\left|\sum_{r=r_2}^{r_3} b_r E_r^{t/k}\sin rx\right| \leq K\sum_{r=r_2}^{r_3} r^{-2}\exp(-2t/r^2k^2). \tag{3.26}$$

The function of r in the last summation assumes its maximum for $r = r_m = \sqrt{4t/3k^2}$. It is an increasing function of r for $r < r_m$, a decreasing one for $r > r_m$, and the value of the maximum is less than $t^{-3/2}k^3$. Therefore, (3.26) can be replaced by

$$\left|\sum_{r=r_2}^{r_3} b_r E_r^{t/k}\sin rx\right| \leq K\int_0^\infty r^{-2}\exp(-2t/r^2k^2)dr + Kt^{-3/2}k^3 \tag{3.27}$$

$$\leq K(k^2/4t + t^{-3/2}k^3) \leq c_{10}t^{-1}Kk^2, \qquad \text{for } t \geq k.$$

On the other hand,

$$\left|\sum_{r=r_3}^{\infty} b_r \exp -r^2t\sin rx\right| \leq K\int_{r_3}^\infty \exp(-r^2t)r^{-2}dr \tag{3.28}$$

$$= -(1/2t)\exp(-r^2tr^{-4})\Big|_{r_3}^\infty - (2/t)\int_{r_3}^\infty \exp(-r^2t)r^{-4}dr \leq t^{-1}Kr_3^{-4} \leq t^{-1}Kk^2,$$

because $r_3 \geq r_0 \geq ((1-s)k)^{-1}$. Combination of (3.27) and (3.28) proves that

$$|V_2| \leq c_{11}t^{-1}Kk^2, \qquad t \geq k. \tag{3.29}$$

d) *The range* $r > r_3$. For $r > r_3 = \pi/h$,

$$|V_3| \leq \sum_{r=\pi/h}^{\infty} |b_r| \leq K\int_{\pi/h}^\infty r^{-2}dr \leq c_{12}Kh^2. \tag{3.30}$$

The desired bound for the truncation error is now obtained by adding the inequalities (3.24), (3.29), and (3.30) to (3.21) or (3.22), respectively. This completes the proof of theorem 2.

The order of magnitude of this appraisal cannot, in general, be improved. One consequence of this theorem is that the particular choice $s = \frac{1}{2}$ improves the order of magnitude of the truncation error only if $h = o(\sqrt{k})$.

As was mentioned before, it is possible to carry out the arguments of this section with the milder hypothesis

$$h^\beta/k \geq p > 0, \qquad \beta > 0,$$

taking the place of (3.2). However, higher powers of t^{-1} will then appear in the bound for the truncation error. If k shrinks so much more slowly than h that

$$|\log|^{-1}/k \leq p < \infty,$$

then the method of proof breaks down decisively.

4. Stability. The term computational stability refers to the extent to which small errors in the data and in the stepwise computation of $U(x, t)$ affect the final result. In the case of linear homogeneous difference equations the cumulative modification of the solution by such errors is the sum of the departures caused by errors on each line $t = 0, k, 2k, \cdots$. Thus, a knowledge of a bound on $U(x, t)$ in terms of the initial function $f(x)$ conveys information concerning the stability of the procedure. In the previous section it was decisive that $f(x)$ had certain smoothness properties. If the round-off errors are to be treated as uniformly bounded, but otherwise arbitrary, quantities, no condition beyond (2.3) and boundedness may be imposed on $f(x)$ in the present section.

In view of the last remark it is natural to base the arguments here on the finite trigonometric interpolation series for $f(x)$, i.e., on

$$f(mh) = \sum_{r=1}^{N-1} B_r \sin mrh, \qquad m = 1, 2, \cdots, N-1, \tag{4.1}$$

where $N = \pi/h$ and

$$B_r = (2/N) \sum_{\rho=1}^{N-1} f(\rho h) \sin \rho rh, \qquad r = 1, 2, \cdots, N-1. \tag{4.2}$$

It can be directly verified that the expression

$$V(mh, nk) = \sum_{r=1}^{N-1} B_r E_r^n \sin mrh \tag{4.3}$$

solves the difference problem (1.3), (1.4) for $U(x, t)$ at the grid points $x = mh$ $t = nk$, $m = 1, 2, \cdots, N-1$; $n = 1, 2, \cdots$, Hence,

$$V(mh, nk) = U(mh, nk).$$

Insertion of (4.2) into (4.3) and an interchange of summations leads to

$$U(mh, nk) = \sum_{\rho=1}^{N-1} g_{\rho mn} f(\rho h) \tag{4.4}$$

with

$$g_{\rho mn} = (2/N) \sum_{r=1}^{N-1} E_r^n \sin mrh \sin \rho rh. \tag{4.5}$$

The quantity (4.5) will now be appraised under the assumption that (2.6) and hence (2.7) is true. Moreover, the restriction

$$0 \leq s \leq 1 \tag{4.6}$$

will be imposed. Under this hypothesis E_r is a decreasing function of r, for $r \leq N-1$. Therefore, E_r^n is a decreasing function of r for all odd n, while for even n the quantity E_r^n decreases as long as $E_r \geq 0$ and increases thereafter, for $r \leq N-1$. In order to take advantage of this observation (4.5) is written in the form

$$g_{\rho mn} = (1/N) \sum_{r=0}^{N-1} E_r^n \cos (m-\rho) rh - (1/N) \sum_{r=0}^{N-1} E_r^n \cos (m+\rho) rh, \tag{4.7}$$

and each of the two right hand terms is subjected to a summation by parts. Recalling the identities

$$\sum_{r=0}^{N-1} a_r b_r = -\sum_{r=0}^{N-1} \left\{ (a_{r+1} - a_r) \sum_{\mu=0}^{r} b_\mu \right\} + a_N \sum_{\mu=0}^{N-1} b_\mu$$

and

$$\sum_{\mu=0}^{r} \cos \mu\theta = \frac{\sin (r+\frac{1}{2})\theta}{2 \sin \frac{1}{2}\theta} + \frac{1}{2} = \frac{\sin ((r+1)\theta/2) \cos (r\theta/2)}{\sin \frac{1}{2}\theta}$$

one finds that

$$\left| (1/N) \sum_{r=0}^{N-1} E_r^n \cos (m-\rho) rh \right| \tag{4.8}$$

$$= \left| -(1/N) \sum_{r=0}^{N-1} \left\{ (E_{r+1}^n - E_r^n) \sum_{\mu=0}^{r} \cos (m-\rho)\mu h \right\} + (1/N) E_N^n \sum_{\mu=0}^{N-1} \cos (m-\rho)\mu h \right|$$

$$\leq \frac{h}{\pi} \frac{\sum_{r=0}^{N-1} |E_r^n - E_{r+1}^n| + |E_N^n|}{|\sin \frac{1}{2}(m-\rho)h|}.$$

Now, if n is odd,

$$\sum_{r=0}^{N-1} |E_r^n - E_{r+1}^n| = \sum_{r=0}^{N-1} (E_r^n - E_{r+1}^n) = E_0^n - E_N^n \leq 2, \qquad \text{odd } n.$$

If E_r is non-negative for $0 \leq r_1 \leq N$ the last formula is correct for even n as well.

But if E_τ vanishes for $= r_0$ one has, instead, with $r_2 = [r_0]$,

$$\sum_{\tau=0}^{N-1} | E_\tau^n - E_{\tau+1}^n | = \sum_{\tau=0}^{[r_0]-1} (E_\tau^n - E_{\tau+1}^n) + | E_{r_2+1}^n - E_{r_2}^n |$$

$$+ \sum_{\tau=r_2+1}^{N-1} (E_{\tau+1}^n - E_\tau^n) = E_0^n - E_{r_2}^n + | E_{r_2+1}^n - E_{r_2}^n | - E_{r_2+1}^n + E_N^n \leq 3,$$

even n.

Inserting the result of the last two inequalities into (4.8) and taking account of the fact that both m and ρ range between 0 and $N = \pi/h$ it follows that the first right hand term in (4.7) is numerically not greater than $4 \mid m - \rho \mid^{-1}$. The second summation in (4.7) is similarly found to be bounded, numerically, by

$$4h/(\pi \mid \sin \tfrac{1}{2}(m + \rho)h \mid),$$

where m and ρ range from zero to π/h. For $m + \rho \leq \pi/h$ that bound is clearly not greater than $4(m + \rho)^{-1}$, which is less than $4 \mid m - \rho \mid^{-1}$. For

$$m + \rho = (2\pi/h) - \gamma, \qquad 0 < \gamma < \pi/h,$$

the bound $4\gamma^{-1} = ((2\pi/h) - m - \rho)^{-1}$ is obtained, instead. But this last quantity is also not greater than $4 \mid m - \rho \mid^{-1}$, since $(2\pi/h) - m - \rho = ((\pi/h) - m) + ((\pi/h) - \rho) \geq \mid ((\pi/h) - m) - ((\pi/h) - \rho) \mid = \mid \rho - m \mid$. Collecting these results it is seen that (4.7) implies the inequality

$$| g_{\rho m n} | \leq \frac{8}{| m - \rho |}.$$

On the other hand,

$$| g_{\rho m n} | \leq 2,$$

because of (2.7) and (4.5). Combination of the last two inequalities yields

$$| g_{\rho m n} | < \frac{10}{| m - \rho | + 1}. \tag{4.9}$$

Let $\bar{f}$ be defined by

$$\bar{f} = \text{l.u.b.}_{0 \leq x \leq \pi} | f(x) |.$$

Then (4.4) and (4.9) imply

$$| U(mh, nk) | \leq 10\bar{f} \sum_{\rho=1}^{N-1} \frac{1}{| m - \rho | + 1}$$

$$\leq 10\bar{f} \left\{ \int_0^m \frac{dx}{1 + m - x} + \int_m^{N-1} \frac{dx}{1 - m + x} + 1 \right\}$$

$$\leq 10\bar{f}(1 + 2 \log (\pi/h)) \leq C\bar{f} \log (1/h),$$

where C is a constant. This proves

THEOREM 3. *If* $f(0) = f(\pi) = 0$, *and* $|f(x)| \leq \bar{f}$ *in* $0 \leq x \leq \pi$; *and if*

$$2(1 - 2s)k - h^2 \leq 0, \qquad 0 \leq s \leq 1,$$

then $|U(mh, nk)| \leq C\bar{f} \log 1/h$, *for* $0 \leq m \leq \pi/h$, $0 \leq n$. *The constant* C *depends on* h_1 *only.*

As a consequence of this theorem the departure caused at a grid point by the cumulative effect of all errors at grid points below that t-level is less than

$$\epsilon\, Ct\bar{f} \log (1/h)k^{-1},$$

where ϵ is an upper bound for the error committed at one point. If h_1 is small C is about equal to 10. Round-off errors of this size are considered quite manageable in numerical work.

It is an open question whether the result of theorem 3 is the best possible, i.e., if there exists a constant C^* such that corresponding to arbitrarily small values of h and k satisfying (2.6) a function $f(x)$ can be found for which

$$|U(mh, nk)| > C^*\bar{f} \log (1/h)$$

at some point $x = mh$, $t = nk$. It is well known that the factor $\log (1/h)$ can be omitted if h, k and s satisfy the inequality

$$s(1 - s)k - h^2 \leq 0,$$

for then the difference equation (1.3) satisfies a maximum principle. But this inequality is more restrictive than (2.6). For $s = \frac{1}{2}$, for instance, it implies the undesirable condition $k/h^2 \leq 1$. The boundedness statement proved in [5] is compatible with values of U of the order $0(h^{-\frac{1}{2}})$ and is therefore weaker than that of theorem 3.

5. A Lemma.

LEMMA: *Let* $\Phi_s(z) = \log \dfrac{1 - (1 - s)z}{1 + sz} + z.$

a) *If* $\min(sz, -(1 - s)z) \geq -\theta_1 > -1$, *then*

$$|\Phi_s(z)| \leq c_1 z^2, \qquad c_1 \text{ a constant.}$$

b) *If* $0 \leq s \leq 1$, $z \geq 0$ and $(1 - s)z < 1$, *then*

$$\Phi_s(z) \leq \tfrac{1}{2}s^2z^2.$$

c) *If* $z/2 \leq \theta_2 < 1$, *then*

$$-c_2z^3 \leq \Phi_{\frac{1}{2}}(z) \leq 0, \qquad c_2 \text{ a constant.}$$

Proof. a) It is readily verified that

$$\Phi_s(z) = \int_{-(1-s)z}^{sz} \frac{\tau}{1 + \tau}\, d\tau, \text{ if } -(1 - s)z > -1, \qquad sz > -1. \tag{5.1}$$

Under the assumption of part a) one has $|1 + \tau| > 1 - \theta_1$ in the interval of integration. Hence,

$$|\Phi_s(z)| \leq (1 - \theta_1)^{-1} \int_{-(1-s)z}^{sz} |\tau|\, d\tau \leq c_1z^2$$

b) Since $-(1-s)z < 0$, $sz > 0$ it follows from (5.1) that

$$\Phi_s(z) \leq \int_0^{sz} \frac{\tau}{1+\tau}\,d\tau \leq \int_0^{sz} \tau\,d\tau \leq \tfrac{1}{2}(sz)^2.$$

c) This result is an immediate consequence of the identity

$$\Phi_1(z) = -2\int_0^{z/2} \frac{\tau^2}{1-\tau^2}\,d\tau.$$

Mathematics Research Center,
United States Army,
University of Wisconsin,
Madison, Wisconsin

1. W. E. MILNE, *Numerical Solutions of Differential Equations*, John Wiley and Sons, Inc., New York, 1953.
2. M. L. JUNCOSA & D. M. YOUNG, "On the order of convergence of solutions of a difference equation to a solution of the diffusion equation," Soc. Indust. Appl. Math., *Jn.* v. 1, 1953, p. 111–135.
3. M. L. JUNCOSA & DAVID YOUNG, "On the convergence of a solution of a difference equation to a solution of the equation of diffusion," Amer. Math. Soc., *Proc.*, v. 5, 1954, p. 168–174.
4. M. L. JUNCOSA & D. M. YOUNG, "On the Crank-Nicolson procedure for parabolic partial differential equations" Cambridge Phil. Soc., *Proc.*, v. 53, 1957, p. 448–461.
5. P. D. LAX & R. D. RICHTMYER, "Survey of the stability of linear finite difference equations," *Comm. Pure Appl. Math.*, v. 9, 1956, p. 267–293.

An Open Formula for the Numerical Integration of First Order Differential Equations (II)

By Herbert S. Wilf

In a previous paper [1], referred to below as *I*, a set of formulas was derived for the numerical integration of systems of first order differential equations. In what follows we will consider the questions of convergence and stability of the method and higher order formulas.

We quote here, for reference, the final results of *I*, namely the propagation formulas:

$$y_1 = y_0 + \frac{h}{12}\left[5f(x_0, y_0) + 8f(x_1, y_1) - f(x_2, y_2^*)\right] \tag{1}$$

$$y_2^* = 5y_0 - 4y_1 + 2h\left[f(x_0, y_0) + 2f(x_1, y_1)\right] \tag{2}$$

for the solution of

$$y' = f(x, y) \tag{3}$$

where y_0 is given.

1. Convergence. The solution of (1) and (2) for y_1 proceeds by taking as an initial guess the y_2^* from the preceding point. One then applies (2) and (1) successively until consecutive values of y_1 agree to sufficient accuracy.

Received 16 August 1957.

The convergence is governed by the following theorem.

THEOREM. *Suppose that on the interval* $[x_0, x_2]$,

$H1$: $\dfrac{\partial f}{\partial y}$ *exists and is everywhere continuous.*

$H2$: $\left|\dfrac{\partial f}{\partial y}\right| \leq M.$

Further suppose that the mesh interval h has been chosen so that

$$H3: \quad 0 < h < \frac{\sqrt{21} - 3}{2M} = \frac{.7926}{M}.$$

Then the iterative process defined by (1) and (2) converges to a solution y_1 of the equations (1), (2).

Proof. Letting y_1 denote the solution of (1), (2) and $y_{1,r}$, $y^*_{2,r}$ denote the r-th iterated values of y_1 and y_2 respectively, we have

$$y_{1,r+1} = y_0 + \frac{h}{12}\left[5f(x_0, y_0) + 8f(x_1, y_{1,r}) - f(x_2, y^*_{2,r+1})\right]$$

and therefore,

$$(4) \quad y_1 - y_{1,r+1} = \frac{h}{12}\{8[f(x_1, y_1) - f(x_1, y_{1,r})] - [f(x_2, y_2^*) - f(x_2, y^*_{2,r+1})]\}.$$

Now,

$$(5) \qquad f(x_1, y_1) - f(x_1, y_{1,r}) = (y_1 - y_{1,r})\frac{\partial f}{\partial y}(x_1, \eta_1)$$

for some η_1, in $(y_1, y_{1,r})$ and

$$(6) \qquad f(x_2, y_2^*) - f(x_2, y^*_{2,r+1}) = (y_2^* - y^*_{2,r+1})\frac{\partial f}{\partial y}(x_2, \eta_2)$$

$$= \frac{\partial f}{\partial y}(x_2, \eta_2)\left[[-4(y_1 - y_{1,r}) + 4h(y_1 - y_{1,r})\frac{\partial f}{\partial y}(x_1, \eta_1)\right]$$

$$= 4(y_1 - y_{1,r})\frac{\partial f}{\partial y}(x_2, \eta_2)\left[h\frac{\partial f}{\partial y}(x_1, \eta_1) - 1\right].$$

Inserting (6), (5) in (4) we get

$$(7) \quad (y_1 - y_{1,r+1}) = (y_1 - y_{1,r})\frac{h}{12}\left\{8\frac{\partial f}{\partial y}(x_1, \eta_1) - 4\frac{\partial f}{\partial y}(x_2, \eta_2)\left[h\frac{\partial f}{\partial y}(x_1, \eta_1) - 1\right]\right\}$$

for some η_2 in $(y_2^*, y^*_{2,r+1})$.

To prove convergence, i.e., that $(y_1 - y_{1,r}) \to 0$, we need only show that the coefficient of $y_1 - y_{1,r}$ in (7) is, in absolute value, < 1. We have

$$|y_1 - y_{1,r+1}| \leq |y_1 - y_{1,r}| \frac{h}{12}\{8M + 4M(hM + 1)$$

$$= |y_1 - y_{1,r}| \left(hM + \frac{h^2M^2}{3}\right).$$

Hypothesis $H3$ assures us that $hM\left(1 + \frac{hM}{3}\right) < 1$ which proves the theorem.

2. Stability. Let y denote the true solution of (3), z denote the calculated solution from (1), (2) and $\eta = y - z$ be the error.

Now y satisfies

$$y_1 = y_0 + \frac{h}{12}[5f(x_0, y_1) + 8f(x_1, y_1) - f(x_2, y_2^*)] + T_1 \tag{8}$$

$$y_2^* = 5y_0 - 4y_1 + 2h[f(x_0, y_0) + 2f(x_1, y_1)] \tag{9}$$

where T_1 is the truncation error given by (8.1) of I, and z satisfies

$$z_1 = z_0 + \frac{h}{12}[5f(x_0, z_0) + 8f(x_1, z_1) - f(x_2, z_2^*)] + h\rho_1 \tag{10}$$

$$z_2^* = 5z_0 - 4z_1 + 2h[f(x_0, z_0) + 2f(x_1, z_1)] + h\rho_2 \tag{11}$$

where ρ_1, ρ_2 are round off error in the evaluation of f.

By subtraction, we get

$$\eta_{n+1}\left\{1 - hg + \frac{h^2g^2}{3}\right\} = \eta_n\left\{1 - \frac{h^2g^2}{6}\right\} + T \tag{12}$$

where we have assumed

a) $\frac{\partial f}{\partial y}$ is constant $(=g)$ over the interval in question.

b) Round off error is negligible.

c) Truncation error is constant $(=T)$.

The solution of (12) is

$$\eta_n = \lambda^n\eta_0 + (\lambda^n - 1)\frac{T}{hg(1 - hg/2)} \tag{13}$$

where

$$\lambda = \frac{1 - (hg)^2/6}{1 - hg + \frac{1}{2}(hg)^2}. \tag{14}$$

The initial error η_0 will therefore decrease in magnitude when $|\lambda| < 1$ which occurs for $-\infty < hg < 0$ and for $hg > 2$.

We may then summarize the discussion of this and the preceding sections by noting that convergence and stability occur simultaneously only when

$$-.7926 < hg < 0. \tag{15}$$

If $\partial f/\partial y$ is positive, errors introduced at any stage will grow in magnitude, and the integration had better be performed in the reverse direction. This behaviour is characteristic of numerical integration techniques.

3. A Higher Order Formula. The formulas corresponding to (1) and (2) for an error of order h^5 are as follows:

$$(16)\quad y_1 = y_0 + \frac{h}{24}\left[9f(x_0, y_0) + 19f(x_1, y_1) - 5f(x_2, y_2^*) + f(x_3, y_3^*)\right] + 0(h^5)$$

$$(17)\quad y_2^* = y_0 + \frac{h}{3}\left[f(x_0, y_0) + 4f(x_1, y_1) + f(x_2, y_2^*)\right]$$

$$(18)\quad y_3^* = 9y_1 - 8y_0 - 3h\left[f(x_0, y_0) + 2f(x_1, y_1) - f(x_2, y_2^*)\right].$$

These formulas are used to find y_1 as follows:

a) Guess $y_1 = y_2^*(x - h)$, $y_2^* = y_3^*(x - h)$.
b) Calculate improved values of y_3^*, y_2^*, y_1 in that order.
c) Repeat from b) until y_1 has converged before proceeding to the next point.

Note that no starter formulas are required since initially we may guess $y_0 = y_1 = y_2^* = y_3^*$ and proceed from b) above.

Nuclear Development Corporation of America,
White Plains, New York

1. HERBERT S. WILF, "An open formula for the numerical integration of first order differential equations," *MTAC*, v. 11, 1957, p. 201–203.

TECHNICAL NOTES AND SHORT PAPERS

Note on the Computation of the Zeros of Functions Satisfying a Second Order Differential Equation

By D. J. Hofsommer

It has been pointed out by P. Wynn [1] that, if a function satisfies a second order differential equation, this fact may be used with advantage in the computation of its zeros. In his note he only pays attention to Richmonds formula which, incidentally, was already known to Schröder [2]. We will elaborate his idea to construct another iteration formula.

Let $f(x)$ be the function, the roots of which are to be computed. Let α be such a root and let x be a first approximation. If the approximation is sufficiently close,

$$(1)\quad \alpha = x - f/f' - \tfrac{1}{2}(f''/f')(f/f')^2 - \tfrac{1}{6}\left[3(f''/f')^2 - f'''/f'\right](f/f')^3 + 0[(f/f')^4].$$

This series may be used either for direct computation in taking enough terms, or for obtaining an iterative process if only few terms are retained. If $f(x)$ satisfies the homogeneous differential equation

$$(2)\qquad f'' = 2Pf' + Qf + 2S,$$

substitution in the series (1) yields

$$(3)\quad \alpha = x - f/f' + (P + S/f')(f/f')^2 - \tfrac{1}{2}(4P^2 - P' + Q + 10PS/f' + S'/f' + 6S^2/f^2)(f/f')^3 + 0[(f/f')^4]$$

Received 3 May 1957.

or,

$$\alpha = x - f/f' - (P + S/f')(f/f')^2 + 0[(f/f')^3] \tag{4}$$

$$= x - \frac{1}{f'/f - P - S/f} + 0\,[(f/f')^3] \tag{5}$$

From (4) and (5) it follows that the iterative processes

$$x_{k+1} = x_k - f(x_k)/f'(x_k) - [P(x_k) + S(x_k)/f'(x_k)][f(x_k)/f'(x_k)]^2 \tag{6}$$

or

$$x_{k+1} = x_k - \frac{1}{f'(x_k)/f(x_k) - P(x_k) - S(x_k)/f'(x_k)} \tag{7}$$

are cubically convergent.

Wynn's process gives

$$x_{k+1} = x_k - \frac{1}{f'(x_k)/f(x_k) - P(x_k) - S(x_k)/f'(x_k) - \frac{1}{2}Q(x_k)f(x_k)/f'(x_k)}$$

if $f(x)$ satisfies (2).

From equality (7) it is seen that the last term in the denominator may be left out without affecting the order of the iteration process.

For comparison we have calculated by aid of (7) the first three zeros of J_0 with one iteration step. The results are

x_0	x_1	x_1 (Wynn)
2.405	2.404 825 557 694	2.404 825 557 697
5.520	5.520 078 110 286	5.520 078 110 286
8.654	8.653 727 912 904	8.653 727 912 914

Of course one iteration step amounts to using the development (4) or (5). If in such a case the required accuracy is almost, but not quite, obtained, application of (3) is preferable to taking a second iteration step.

In such a way we have calculated the zeros of $P_{37}(\cos\varphi)$. Using Smith's table [3] the zeros φ_m were found in hundredths of a degree. Then new values for φ_m were calculated by aid of (3) or (4) using the NBS tables [4]. In this way interpolation was avoided. We found

m	φ_m(radians)	m	φ_m(radians)
		9	0.816 897 487 6
0	1.570 796 326 8	10	0.733 136 902 8
1	1.487 027 983 4	11	0.649 379 438 3
2	1.403 259 745 5	12	0.565 626 516 9
3	1.319 491 725 5	13	0.481 880 536 8
4	1.235 724 047 9	14	0.398 145 883 1
5	1.151 956 859 3	15	0.314 431 540 1
6	1.068 190 338 7	16	0.230 759 304 7
7	0.984 424 715 0	17	0.147 197 715 6
8	0.900 660 291 7	18	0.064 126 781 0

The approximation in hundredths of a degree could have been obtained also by

starting with the approximation for the zeros of $P_n(\cos\varphi)$

$$\varphi_m \doteq \left(\tfrac{1}{2} - \frac{2m}{2n+1}\right)\pi, \qquad n \text{ odd}$$

$$\varphi_m \doteq \left(\tfrac{1}{2} - \frac{2m+1}{2n+1}\right)\pi, \qquad n \text{ even}$$

and applying one Newton step $x_{k+1} = x_k - f(x_k)/f'(x_k)$.

Finally we remark that if $f(x)$ is a member of a set $f_n(x)$, it often happens that $f_n'(x)$ satisfies a recurrence relation

$$f_n'(x) = A(x)f_{n-1}(x) + B(x)f_n(x).$$

It then may be useful to transform

$$f_n'(x_k)/f_n(x_k) = A(x_k)f_{n-1}(x_k)/f_n(x_k) + B(x_k),$$

especially if $f_n(x)$ is tabulated and $f_n'(x)$ not or if the tables of $f_n(x)$ are more extensive than those of $f_n'(x)$.

Mathematical Centre,
Computation Department,
Amsterdam, Netherlands

1. P. Wynn, "On a cubically convergent process for determining the zeros of certain functions," *MTAC*, v. 10, 1956, p. 97–100.
2. E. Schröder, "Ueber unendlich viel Algorithmen zur Auflösung der Gleichungen," *Math. Ann.*, v. 2, 1870, p. 317–363.
3. E. R. Smith & Archie Higdon, "Zeros of the Legendre polynomials," *Jn. Science*, Iowa State College, v. 12, 1938, p. 263–274." (*MTAC*, Rev. 132, v. 1, 1944, p. 153–514.)
4. NBSCL, *Table of Sines and Cosines to Fifteen Decimal Places at Hundredths of a Degree* (*NBS Applied Math. Series*, no. 5), Govt. Printing Office, Washington, 1949. (*MTAC*, Rev. 662, v. 3, 1949, p. 515–516.)

A New Mersenne Prime

by Hans Riesel

On September 8, 1957, the Swedish electronic digital computer BESK established that the Mersenne number $M_{2217} = 2^{2217} - 1$ is a prime. The result was recalculated on September 12. The method used for testing this number was a Lucas' test, and the running time was $5^h\ 30^m$. The new prime has 969 decimal digits and on the BESK was found to be

$$2^{2217} - 1$$

2	591 17086	013 20262	777 62467	679 22441	530 94181	888 75531	254 27303
974 92316	187 40192	665 86362	086 20120	951 68004	834 06550	695 24173	319 41774
416 89509	238 80701	741 03777	095 97512	042 31306	662 40829	163 53517	952 31118
615 48622	656 04547	691 12759	584 87756	105 68757	931 19101	771 14088	262 52153
849 03583	040 11850	721 16424	747 46182	303 14713	983 40229	288 07454	567 79079
410 37288	235 82070	589 23510	684 33882	986 88861	665 86502	809 27692	080 33960
586 93087	905 00409	503 70987	590 21190	183 71991	620 99400	256 89351	131 36548
829 73911	265 67973	032 41986	517 25011	641 27035	097 05427	773 47797	234 98216
764 43446	668 38311	932 25400	996 48994	051 79024	162 40565	190 54483	690 80961
606 16257	430 42361	721 86333	941 58524	264 31208	737 26659	196 20617	535 35748
892 89459	962 91951	830 82621	860 85340	093 79328	394 20261	866 58614	250 32514
507 73096	274 23537	682 29386	494 07127	700 84607	712 42118	230 80804	139 29808
705 75047	138 25264	571 44837	937 11250	320 81826	126 56664	908 42516	994 53951
887 78961	365 02484	057 39378	594 59944	433 52311	882 80123	660 40626	246 86092
121 50349	937 58478	229 22371	443 39628	858 48593	821 57388	212 32393	687 04616
067 73629	093 15071						

Received 8 November 1957.
Ormängsgatan 67,
Stockholm, Sweden

REVIEWS AND DESCRIPTIONS OF TABLES AND BOOKS

1[C, D].—J. PETERS, *Ten-place logarithm table:*

v. 1. Ten-place logarithms of numbers from 1 to 100,000, together with *Appendix, Mathematical Tables*, edited and computed by J. Peters and J. Stein (with *22-place logarithms of the trigonometric functions* by G. Witt),

v. 2. *Ten-place logarithms of the trigonometric functions from* 0° *to* 90° *for every Thousandth of a Degree,*

Auxiliary Tables for Ten-place logarithms table,

Published by the Reich Office for Land Survey by J. Peters, Under Scientific Supervisor, Berlin, 1922, 1919, 1919, New Revised Edition, with English translations, F. Ungar Publishing Co., New York, 1957, xvi + 607 + xxciii + 195, vii + 901, 76 + 53 p., 27 cm. Price $50.00 per set.

This is a very welcome (photographic) reprint of volumes long out of print, which have been much sought after, perhaps more for the Appendix, familiarly known as Peters-Stein, than for the main tables. The reproduction is excellent, the binding seems durable, and there is an English translation by C. J. Hyman of the German introductory and explanatory text. A supplementary list of the errors which have been noticed since the original printing has been prepared by C. J. Hyman; there is even a translation of this into German.

The reviewer would like to point out that three errors which are associated with him (p. 51 of the Auxiliary volume) were actually found by the late C. R. Cozens and by Dr. J. W. Wrench, Jr. They were originally pointed out in this journal [1], and this fact was noted by the reviewer in his communication with the publisher.

According to R. C. Archibald [2], J. T. Peters (1869–1941) was perhaps the greatest mathematical table maker of all time. An astronomer by training, he was largely responsible for the official German nautical and astronomical tables. In addition, he prepared a long series of mathematical tables, at varying numbers of decimal places, all of the highest standard of accuracy and of format.

For those who want a set of logarithmic tables, of upper middle class precision, with trigonometrical functions for arguments in degrees and decimals, the present volumes are unequalled. They include the following:

In volume I there is a table of logarithms to 10*D* of the integers from 1 to 1,000 and from 10,000 to 100,000, occupying about 600 pages. First differences are tabulated.

In volume II there is a 10*D* table with positive characteristics, of log sin x, log tan x, log cot x, log cos x, for $x = 0(.001)45°$. First differences are tabulated, except for small angles, where other methods of interpolation are to be used (see b) below). These occupy 900 pages.

In the third volume there are auxiliary tables for use with 1), 2), which occupy about 70 pages. This volume also includes the translations of the textual matter, which occupies 53 pages.

a) Tables for improvement of first difference, given the second difference (0(10)1000) and the phase (0(.01)1) (20 pages).

b) Tables of $S = \log \sin x - \log x$ and $T = \log \tan x - \log x$, with first differences, for $x = 0(.001)2.1°$, to be used for interpolation (21 pages).

c) Miscellaneous relevant conversion tables (22 pages).

The various introductions give some account of the sources and construction and checking of the tables; there are worked examples to show the uses of the main and auxiliary tables. Included are Grimpen's values of $\log_{10} p$ to $84D$ and $\log_e p$ to $82D$, for primes up to 113.

In addition to these tables there is, in Volume I, an Appendix (of about 200 pages) containing the basic numerical data which Peters and his collaobrators used in their work. This collection, made in collaboration with J. Stein (which is supplemented by a $22D$ table of the logarithms of the trigonometrical functions due to G. Witt), is invaluable to all serious computers. We shall describe its contents in some detail.

1) Many decimal values of such constants as π, e, γ, $M = \log_{10} e$, together with various powers and multiples (12 pages).

2) Exact values of the first ten powers of the integers up to 308 with certain higher powers of the primes under 100 (23 pages).

3) Reciprocals, to $32D$, of the first sixteen powers of the integers up to 100 (22 pages).

4) Exact values of $n!$ for n up to 60, and their expressions in prime factors (2 pages).

5) Reciprocals, to $54D$, of $n!$ for n up to 43. Also e, e^{-1}, sin 1, cos 1, sinh 1, cosh 1 to $52D$ (1 page).

6) Common logarithms of $n!$, to $18D$, for n up to 1200 (8 pages).

7) Exact values of binomial coefficients $\binom{n}{k}$ for n up to 60, $0 \leq k \leq n$, and their expressions in prime factors (14 pages).

8) Exact values (as rational fractions in lowest terms) of the Bernoulli numbers B_n, for n up to 90. Also $\log_{10} B_n$ to $10D$ and B_n/B_{n-1} to $4S$ for n up to 250 (5 pages).

9) Exact values of the tangent numbers T_n and the Euler numbers E_n, for n up to 30. Also $\log_{10} T_n$ and $\log_{10} B_n$ to $10D$, T_n/T_{n-1} and B_n/B_{n-1} to $4S$, for n up to 50 (2 pages).

10) Various sums of reciprocals, all to $32D$: $2^{-n} \pm 3^{-n} + 4^{-n} \pm 5^{-n} + \cdots$ and $2^{-n} + 4^{-n} + 6^{-n} + \cdots$ up to $n = 100$; $4^{-n} + 6^{-n} + 8^{-n} + \cdots$ and $3^{-n} + 5^{-n} + 7^{-n} + \cdots$ up to $n = 50$; $3^{-n} - 4^{-n} + 5^{-n} - 6^{-n} + \cdots$ and $3^{-n} - 5^{-n} + 7^{-n} - 9^{-n} + \cdots$ up to $n = 53$ (5 pages).

11) Early coefficients, to $24D$, of terms in power series for direct and logarithmic trigonometrical functions of $\frac{1}{4}\pi x$ (or related auxiliary functions such as $\tan \frac{1}{4}\pi x - 4/\pi\{(2 - x)^{-1} - (2 + x)^{-1}\}$) (3 pages).

12) Complete factorization of integers up to 10192 (29 pages).

13) Various many decimal natural logarithm tables: $\log_e n$, to $48D$, for integers up to 146 and thereafter for primes up to 9973; radix table of $\log_e (1 \pm a \cdot 10^{-b})$, to $48D$, for $a = 1(1)9$, $n = 4, 5, 6$; $\log_e 2$, $\log_e 3$, $\log_e 5$, $\log_e 7$ to $272D$ (26 pages).

14) Various many-decimal common logarithm tables: radix table of $\log_{10} (1 \pm a \cdot 10^{-n})$, to $28D$, for $a = 1(1)9$, $n = 1(1)14$; $\log_{10} n$ for $n = 2(1)9$, to $28D$;

$\log_{10} n$, to 61*D*, for integers n up to 100 and thereafter for primes up to 1097; radix table of $\log_{10}(1 \pm a \cdot 10^{-n})$, to 61*D*, for $a = 1(1)9$, $n = 3, 4, 5, 6$ (10 pages).

Witt's Tables. I. $\log \sin \alpha$, $\log \cos \alpha$, $\log \cos \alpha$, $\log \tan \alpha$, to 22*D*, for $\alpha = 0(10')45°$ (positive characteristics).

II. $\log \sin \beta$, $\log \cos \beta$, $\log \tan \beta$, for $\beta = 0(1'')10'$ and $T = \log \tan \beta - \log \beta''$ together with the first three differences of $\log \cos \beta$ and T. It is shown how to get 21*D* values of the trigonometrical functions and their logarithms, at $1''$ interval, using these tables and the addition formulae (33 pages).

JOHN TODD

California Institute of Technology
Pasadena, California

1. Errors noted by J. W. WRENCH, *MTAC*, v. 2, 1946, p. 138 and v. 6, 1952, p. 255, and by C. R. COZENS, v. 1, 1943, p. 57.
2. RAYMOND CLARE ARCHIBALD, "Mathematical Table Makers. Portraits, Paintings, Busts, Monuments, Bio-Bibliographical Notes," Scripta Math. Studies, No. 3, *Scripta Mathematica*, New York, 1948, p. 68.

2[F].—A. GLODEN, *Table de factorisation des nombres* $N^4 + 1$ *dans l'intervale* $20000 < N \leq 30000$, 100 p., 27 cm., hand-written, placed in the UMT FILE.

[F]—.A. GLODEN, *Table de factorisation des nombres* $N^4 + 1$ *dans l'intervalle* $30000 < N \leq 40000$, 100 p., 29 × 21 cm., deposited in the UMT FILE.

These manuscripts considerably extend the tables of factorizations of $N^4 + 1$. All odd prime factors less than 800 000 are listed. If the remaining factor is less than 64×10^{10} it is prime and the factorization is complete. The primes greater than 800 000 thus found are listed in the volume $30\,000 < N \leq 40\,000$. Factorizations of $N^4 + 1$ have been published for $N < 6000$, with many entries blank or incomplete. [For $n \leq 1000$, see *MTAC*, v. 2, 1947, p. 252. For $1000 < N \leq 3000$ see *MTAC*, v. 2, 1947, p. 211; v. 3, 1948, p. 118. For $3000 < N \leq 6000$ see *MTAC* v. 4, 1950, p. 224; v. 11, p. 274. For $6000 < N \leq 10\,000$ see *MTAC*, v. 5, 1951, p. 28. For $10\,000 < N \leq 20\,000$ see *MTAC*, v. 9, 1955, p. 26. For $30\,000 < N \leq 40\,000$ see *MTAC*, v. 6, 1952, p. 102; v. 7, 1953, p. 33.]

J. L. SELFRIDGE

University of California
Los Angeles, California

3[F].—A. SCHOLZ & B. SCHOENEBERG, *Einführung in die Zahlentheorie*, Sammlung Göschen Band 1131, Walter de Gruyter and Co., Berlin, 1955, 128 + 20 p. 15 cm. Price DM 2.40.

This is a second edition of the booklet of the same title, by A. Scholz alone, which was first published in 1939, and which is a compact source of information in the subject of elementary number theory. Scholz died in 1942 and the new edition was prepared by B. Schoeneberg. In the new version some misprints and errors have been corrected and the material has been brought up-to-date, e.g., recent numerical work on Mersenne numbers has been included. However, the chapter on "Algorithmisches Rechnen" has been omitted altogether; this is a great loss to

those interested in computation, for it contained accounts of many useful ideas in this area, in which Scholz was a real expert. The booklet is divided into four chapters: Divisibility properties; Congruences and residue classes; Quadratic Residues; Quadratic Forms.

OLGA TAUSSKY

California Institute of Technology
Pasadena, California

4[G].—EMMA LEHMER, *Roots of the reduced quintic, sextic, and seventh degree period equations*, 22 p., 37 × 28 cm., tabulated from cards punched by SWAC. Deposited in the UMT FILE.

These tables list the generalized Gauss sum $S_k = \sum_{m=0}^{p-1} \exp(2\pi i m^k/p)$ for each prime $p = kf + 1$. This sum is the principal root of the reduced period equation of degree k. The other $k - 1$ roots are computed using similar sums of cyclotomic cosines [1], and are listed. The tables extend for $k = 5$ to $p = 10061$, for $k = 6$ to $p = 4093$ and for $k = 7$ to $p = 5153$. The evidence for and against various conjectures about the distribution of S_k is discussed in [1].

J. L. SELFRIDGE

University of California
Los Angeles, California

1. EMMA LEHMER, "On the location of Gauss sums," *MTAC*, v. 10, 1956, p. 194–202.

5[K].—J. AITCHISON & J. A. C. BROWN, *The Lognormal Distribution*, University of Cambridge, Department of Applied Economics Monographs, 5, Cambridge University Press, Cambridge, 1957, xviii + 176 p., 25 cm. Price $6.50.

This useful monograph gives a thorough discussion of the history, theory, and applications of the lognormal frequency function, bringing together material hitherto to be found only in widely scattered publications. Though the authors' principal motivation lay in applications to economic research, applications to other fields have not been slighted. Four new tables are contained in the appendix. The first gives the coefficient of variation, the skewness, the kurtosis, the ratio of the mean to the median, the ratio of the mean to the mode, the frequency up to the mean, and the Lorenz measure of concentration all to 4*D* or 4*S* for the standard deviation, σ, of the normally distributed variable = 0(.05)1(.1)4. The next two tables give values of two functions defined by the series,

$$\psi_n(t) = 1 + \sum_j \frac{(n-1)^{2j} t^j}{n^j (n-1)(n+1)\cdots(n-3+2j) j!}$$

and

$$x_n(t) = \psi_n(2t) - \psi_n\left(\frac{n-2}{n-1} t\right)$$

which are required in finding minimum variance unbiased estimates of the mean and variance of the lognormal distribution from a sample of n. These are tabulated to 4*D* for $n = 10\ (10)\ 200\ (100)\ 500, 750, 1000, 5000, \infty$ for both functions; for $t = .05\ (.05)\ 2$ for $\psi_n(t)$, and for $t = .05\ (.05)\ 1$ for $\chi_n(t)$. The fourth table lists

solutions of the equation $u^3 + 3u = k$, required for moment estimation in the three parameter case, to $4D$ for $k = 0(.2)10(1)24$. The corresponding estimates of the variance are also given to $4D$. An additional pair of tables gives for 65 artificial samples of sizes 32, 64, 128, 256, and 512 the results of various methods of estimation of the mean and variance.

C. C. C.

6[K].—SIDNEY SIEGEL, *Nonparametric Statistics for the Behavioral Sciences* McGraw-Hill Book Co., New York, 1956, ix + 312 p., 23.7 cm. Price $6.50.

This useful text contains twenty-one tables in an appendix. None of these are new but, like the text which brings together in one place discussions and illustrations of a good selection of the more important nonparametric tests which previously had been available from rather scattered sources, this appendix makes the associated tables readily available. The sources from which these tables are either copied or adapted are given in each case.

In addition to tables of areas under the normal frequency curve, percentage points of the Student-Fisher t and χ^2, factorials, binomial coefficients and squares and square roots of the kind ordinarily found in statistics texts, the following are more special:

Table D. The cumulative binomial probabilities,

$$\sum_{r=0}^{x} \binom{N}{r} (\tfrac{1}{2})^N \text{ to } 3D \text{ for } x = 0(1)N \text{ and } N = 5(1)25.$$

Table E. 1, 5, 10, 15 and 20 percentage points for the Kolmogorov-Smirnov one-sample statistic D to $3D$ for $N = 1(1)20$, to $2D$ for $N = 25, 30, 35$ with a large sample approximation formula for $N > 35$.

Table F. Upper and lower 5% values for the number of runs in sequences of alternatives for the numbers of the two kinds of elements, $n_1, n_2 = 2(1)20$.

Table G. Critical values of T in the Wilcoxon matched-pair signed-ranks test at the .025, .01 and .005 levels for the one-tailed test (.05, .02 and .01 for the two-tailed test) for the number of matched pairs $N = 6(1)25$.

Table H. A page of the tests on the median due to J. E. Walsh with one and two-tailed significance levels to 3D for sample sizes $N = 4(1)15$.

Table I. For the 2×2 contingency table with row entries A, B and C, D using the Fisher-Yates test, .05, .025, .01, and .005 critical values for C given A (or for D given B) for $C + D = 2(1)A + B$ and $A + B = 3(1)15$.

Table J. Probabilities to 3D of values of U as small as that observed in the Mann-Whitney test for the sample sizes $n_1 \leq n_2 = 3(1)8$.

Table K. Critical values of u in the Mann-Whitney test for the significance levels .001, .01, .025, .05 for the one-tailed test (.002, .02, .05, .1 for the two-tailed test) for $n_1 = 1(1)20$ and $n_2 = 9(1)20$.

Table L. Critical values of K_d in the Kolmogorov-Smirnov two-sample test at the .01 and .05 levels for the one-tailed and two-tailed test for the sample sizes $n_1 = n_2 = 3(1)30, 35, 40$.

Table M. Approximate formulae for critical values of D in Kolmogorov-Smirnov

two-sample test at the .001, .005, .01, .025, .05, and .1 significance levels for the two-tailed test for large samples.

Table N. Probabilities to 3 or more D for χ_r^2 as large as that observed in the Friedman two-way analysis of variance by ranks for N sets of K matched samples. For $K = 3$ and $N = 2(1)9$ values are given for selected values of x_r^2 covering the effective range in each case. For $K = 4$ values are given for similarly selected χ_r^2's for $N = 2, 3, 4$.

Table O. Probabilities to 3D for selected values of H as large as that observed in the Kruskal-Wallis one-way analysis of variance by ranks for all sets of sample sizes n_1, n_2, n_3 up to $n_1 = n_2 = n_3 = 5$.

Table P. Critical values of the Spearman rank correlation coefficient to 3D at the .01 and .05 levels for the one-tailed test for sample sizes, $N = 4(1)10(2)30$.

Table Q. Probabilities to 3D or 2S for values of the Kendall rank correlation coefficient S as large as that observed for $S = 0(2)36$ with sample size $N = 4, 5, 8, 9$ and for $S = 1(2)45$ with $N = 6, 7, 10$.

Table R. Critical values of s to 1D in the Kendall coefficient of concordance at the .01 and .05 significance levels for k sets of N rankings. For $N = 3$, values are given for $k = 8, 9, 10, 12, 14, 15, 16, 18, 20$; for $N = 4$ for $k = 4, 5, 6, 8, 10, 15, 20$; for $N = 5, 6, 7$ for $k = 3(1)6, 8, 10, 15, 20$.

C. C. C.

7[K].—NBS Applied Mathematics Series, No. 48, *Fractional Factorial Experiment Designs for Factors at Two Levels*, U. S. Gov. Printing Office, Washington, D. C., 1957, iv + 85 p., 26 cm. Price $.50.

This table gives designs for experiments with from 5 to 16 factors, each of which may occur in two levels. These involve fractional replication ranging from 1/2 to 1/256. The designs are given in terms of blocks. For each design we are told which main effects are measurable, which two-factor interactiona are measurable for blocks only, and also for blocks and rows. In addition we are told which combinations are block confounding and which are row confounding. The number of designs given is so large that it is to be expected that an experimenter will find exactly the design needed for any desired information on two level factors involving not more than 16 factors.

MARSHALL HALL, JR.

The Ohio State University
Columbus, Ohio

8[K].—D. TEICHROEW, "Tables of expected values of order statistics and products of order statistics for samples of size twenty and less from the normal distribution," *Ann. Math. Stat.*, v. 27, 1956, p. 410–426.

These tables represent the end product of a project undertaken originally by the Institute for Numerical Analysis, National Bureau of Standards, to extend, both in terms of sample size and decimal place accuracy, existing tables of the means, variances, and covariances of order statistics for samples from a normal population. Tables I and II give expected values of the order statistics and their products to 10D for samples of size $n = 1(1)20$. Tables III and IV give certain

auxiliary functions required in the calculation, and in any further extension to larger sample sizes, to 25D. Tables I and II to 20D can be obtained on 925 punched cards by writing the Director, Numerical Analysis Research, University of California, Los Angeles 24, California, and requesting table 5024. A charge of three new cards for each one used will be made.

C. A. BENNETT

Hanford Laboratories Operation
General Electric Company
Richland, Washington

9[K].—P. K. BHATTACHARYA, "Joint test for the mean and variance of a normal population," Calcutta Stat. Assn., *Bull.*, v. 6, 1955, p. 73–90.

Let $x_1, \cdots, x_n$ be a sample from $N(m, \sigma^2)$ and let $n\bar{x} = \sum x_i$, $ns_0^2 = \sum (x_i - m_0)^2$, $(n-1)s^2 = \sum (x_i - \bar{x})^2$, $st = (\bar{x} - m_0)\sqrt{n}$. The problem of testing simultaneously $H_1: m = m_0$ against $m > m_0$, and $H_2: \sigma = \sigma_0$ against $\sigma > \sigma_0$ is regarded as a four-decision problem. We reject H_1 when $\bar{x} > c$, and H_2 when $s^2 > k$, where c and k are fixed by requiring that Pr (reject H_1) $= Pr$ (reject H_2) and Pr (accept both) $= 0.95$, when $m = m_0$ and $\sigma = \sigma_0$. A second procedure is obtained by replacing $\bar{x}$ by t and s^2 by s_0^2. For both procedures we are given to 3D the probabilities of the four decisions, for $n = 5(5)20$, $\sigma^2/\sigma_0^2 = 1(.5)3.5$, and $\sqrt{n}(m - m_0) = 0, .1, .4, 1$, and 1.5.

J. L. HODGES, JR.

University of California
Berkeley, California

10[K].—P. B. PATNAIK, "Hypotheses concerning the means of observations in normal samples," *Sankhyā*, v. 15, 1955, p. 343–372 (appendix).

The non-central t' distribution is given by

$$p(t', \nu, \delta) = C(\nu)\left(\frac{\nu}{\nu + t'^2}\right)^{\nu+1/2} Hh_\nu\left(-\frac{t'\delta}{(\nu + t'^2)^{1/2}}\right)$$

where $Hh_\nu(x) = \int_0^\infty \frac{u^\nu}{\nu!} \exp \tfrac{1}{2}(u + x)^2 du$. The solution of $P(t' \leq x) = .05$, $P(t' \geq x) = .05$ is tabulated to 2D for $\nu = 4, 5$, $\delta = 0(.5)4$, and to 3D for $\nu = 6(1)15, 20$, $\delta = 0(.5)4$. The tabulation is based on the tables by Johnson and Welch [1]. An approximation for the upper and lower percentage points is given in terms of the central t-distribution. A number of statistical hypotheses for which the t' statistic can be used are discussed.

INGRAM OLKIN

Michigan State University
East Lansing, Michigan

1. N. L. JOHNSON & B. L. WELCH, "Applications of the non-central t-distribution," *Biometrika*, v. 31, 1940, p. 362–389.

11[K].—F. N. DAVID & N. L. JOHNSON, "Some tests of significance with ordered variables," Roy. Stat. Soc., *Jn.*, ser. B, v. 18, 1956, p. 1–31.

A survey of test procedures based on medians, quartiles, interquartile ranges, etc., is made. Numerous small tables, computed by the method of moments, of the approximate percentage points are presented. In particular, to test the hypothesis of equality of the means of two normal universes with equal but unknown variance, the test criterion $T = (X_{M_1} - X_{M_2})(N_1 + N_2)/N_1(X_{u_1} - X_{l_2}) + N_2(X_{u_2} - X_{l_2})$ is proposed in which N_1, N_2; X_{M_1}, X_{M_2}; X_{u_1}, X_{u_2}; X_{l_1}, X_{l_2} are the sample sizes, the medians, and the upper and lower quartiles in samples from the two universes respectively. 1%, 2.5%, and 5% points for T are given to 3D for N_1, $N_2 = 11(4)51$.

I. R. SAVAGE

Center for Advanced Study
in the Behavioral Sciences
Stanford, California

12[K].—C. I. BLISS, W. G. COCHRAN, & J. W. TUKEY, "A rejection criterion based upon the range," *Biometrika*, v. 43, 1956, p. 418–422.

The data consists of k independent samples of n measurements each. Under the null hypothesis, the measurements are assumed to be normally distributed and have the same variance. The alternative hypothesis of interest is that one of the measurements is too large or too small to be consistent with the null hypothesis. That is, the problem is that of rejecting an outlying observation. To perform the proposed test first compute the range for each sample. Let T equal the largest of the k ranges divided by the sum of all the ranges. If T is significantly large, the null hypothesis is rejected and the aberrant observation determined by inspection and rejected. Table 1 contains 3D upper 5 per cent points of T for $k = 2(1)10$ and $n = 2(1)10$. Table 2 contains 2D upper 5 per cent points of $(k + 2)\ T$ for $k = 10, 12, 15, 20, 50$, and $n = 2(1)10$.

J. E. WALSH

Military Operations Research Division
Lockheed Aircraft Corporation
Burbank, California

13[K].—H. A. DAVID, "Revised upper percentage points of the extreme studentized deviate from the sample mean," *Biometrika*, v. 43, 1956, p. 449–451.

The upper 100 α percentage points of the extreme studentized deviate in a random sample of n observations, based on an independent estimate of the variance with ν degrees of freedom, are given to 2D for $\alpha = .10, .05, .025, .01, .005$; and to 1D for $\alpha = .001$, for $n = 3(1)12$ and $\nu = 10(1)20, 24, 30, 40, 60, 120, \infty$. Accuracy to one unit in the last place recorded is claimed. This is a revision and slight extension of the original tables by Nair [1].

C. A. BENNETT

Hanford Laboratories Operation
General Electric Company
Richland, Washington

1. K. R. NAIR, "Tables of percentage points of the 'Studentized' extreme deviate from the sample mean," *Biometrika*, v. 39, 1952, p. 189–191. [*RMT* 1182, *MTAC*, v. 8, 1954, p. 84.]

14[K].—CHARLES W. DUNNETT, "A multiple comparison procedure for comparing several treatments with a control," Amer. Stat. Assn., *Jn.*, v. 50, 1955, p. 1096–1121.

This paper is concerned with the case where the numerical results of an experiment performed to compare p treatments with a control can be summarized in the form of a set of numbers $\bar{x}_0, \bar{x}_1, \cdots, \bar{x}_p$, and s, where the $\bar{x}$'s are the means of the $p + 1$ sets of observations which are assumed to be independently and normally distributed, $\bar{x}_0$ referring to the control, and s is an independent estimate of the common standard deviation. A procedure is presented for making confidence statements about the true values of the p differences $\bar{x}_i - \bar{x}_0$, the procedure having the property that the probability of all p statements being simultaneously correct is equal to a specified value P. Tables are available that enable one to use the procedure for $P = .95$ or $.99$, $p = 1(1)9$, and degrees of freedom $= 5(1)20, 24, 30, 40, 60, 120, \infty$. When the number of observations in each set are equal the tables yield exact probabilities for the 3D values given for either one-sided or two-sided confidence limits while if the number of observations are not equal the value of P must be approximated. Illustrative examples are given along with a theoretical discussion of the procedure. A section dealing with the problem of optimum allocation of observations between control and treatments is also included in the paper.

C. F. KOSSACK

Purdue University
Lafayette, Indiana

15[K].—GERALD J. LIEBERMAN & GEORGE J. RESNIKOFF, "Sampling plans for inspection by variables," Amer. Stat. Assn., *Jn.*, v. 50, 1955, p. 457–516.

Inspection procedures are known as sampling inspection by variables when items of a sample drawn at random from a lot are classified into defective or nondefective on the basis of a variable quality characteristic; the decision to accept or reject the lot is a function of these measurements (as opposed to the decision being based on the number of defectives as in sampling by attributes). Measurements of this quality characteristic are assumed to be independent, identically distributed normal random variables.

Sampling plans are presented giving the sample size n and the maximum allowable estimated proportion of defectives p. They are classified according to Acceptable Quality Level (AQL) and sample size. Tables are presented for the purpose of transforming the data into uniform minimum variance unbiased estimates of the percentage defective in the lot.

Table I presents inspection plans based on a known standard deviation for AQL's $= .04, .065, .1, .15, .25, .4, .65, 1, 1.5, 2.5, 4, 6.5, 10, 15$ all in per cents. Minimum values of $(U - L)/\sigma$ are given below which no inspection is necessary in order to reject the lot. U and L denote upper and lower specifications of the quality characteristic and σ denotes the known standard deviation. Table II gives the estimates of the lot percentage defective to 3D for

$$C_U' = \frac{U - \bar{X}}{\sigma}\left(\frac{n}{n-1}\right)^{\frac{1}{2}} \text{ and } C_L' = \frac{\bar{X} - L}{\sigma}\left(\frac{n}{n-1}\right)^{\frac{1}{2}} = 0(.01)4$$

in which σ is the known standard deviation. Table III gives sampling plans based on an unknown standard deviation for the same set of AQL's and Table IV gives estimates of the lot percentage defective to 2D for $C_U = (U - \bar{X})/S$ and $C_L = (L - \bar{X})/S = 0(.1)3.9$ where S^2 is the unbiased estimate of the variance. Tables V and VI are strictly analogous tables for use with the average range, $\bar{R}$, the arguments in Table VI being $C_U'' = k(U - \bar{X})/\bar{R}$ and $C_L'' = k(\bar{X} - L)/\bar{R}$ with the values of k given to 3D in Table V for each sampling plan. Figures 1 up to 16 present OC curves for all the sampling plans based on unknown standard deviation (curves for sampling plans based on average range and known standard deviation are essentially equivalent).

CYRUS DERMAN

Columbia University
New York, N. Y.

16[K].—STANLEY REITER, "Estimates of bounded relative error for the ratio of variances of normal distributions," Amer. Stat. Assn., *Jn.*, v. 51, 1956, p. 481–488.

Let n_1, n_2 be sizes of independent random samples, respectively. Let $\theta = n_1/n_2$. Enter Table 1 with $\theta = 1, 1.5, 2, 5, 10, 100$, and for a required bounded relative error K (in the sense of Blackwell and Girshick [1] $K = 1.01, 1.02, 1.05, 1.10, 1.25, 1.30, 1.40, 1.50$) find β^* to 4D to be used in the "best" estimate $\beta^*(\hat{\sigma}_1^2/\hat{\sigma}_2^2)$ of σ_1^2/σ_2^2, where $n_1\hat{\sigma}_1^2/\sigma_1^2$ and $n_2\hat{\sigma}_2^2/\sigma_2^2$ are required to be independently distributed as χ^2 with n_1 and n_2 degrees of freedom. The confidence coefficient α associated with the estimate is then read from one of seven charts which show α as a function of n_1, n_2, and K (K as above, excluding $K = 1.01$).

H. A. FREEMAN

Massachusetts Institute of Technology
Cambridge, Mass.

1. DAVID BLACKWELL & M. A. GIRSHICK, *Theory of Games and Statistical Decisions*, John Wiley and Sons, Inc., New York, 1954, p. 316–323.

17[K].—EDGAR J. GILBERT, "The matching problem," *Psychometrika*, v. 21, 1956, p. 253–266.

Each of two decks of cards consists of s suits of c cards per suit. One deck is thoroughly shuffled, and each card is then compared with the card in the same ordinal position in the other deck. If the two cards compared are of the same suit a "matching" is recorded. The probability of h or more matchings is tabulated to 5D for $s = 2(1)11$ for $c = 1, 2$; for $s = 2(1)8$ for $c = 3$; for $s = 2(1)5$ for $c = 4, 5$; for $s = 2, 3$ for $c = 6, 7$; and for $s = 2$ for $c = 8(1)12$; in each case for all possible values of h. The fitting of various standard statistical distributions as an approximation to other cases is discussed. It is concluded that the normal distribution is frequently wide of the mark, and that the Gram-Charlier type B is the most generally accurate.

T. N. E. GREVILLE

Social Security Administration
Washington, D. C.

18[K].—A. CLIFFORD COHEN, JR. "Maximum likelihood estimation of the dispersion parameter of a chi-distributed radial error from truncated and censored samples with applications to target analysis," Amer. Stat. Assn., *Jn.*, v. 50, 1955, p. 1122–1135.

The radial error (from the true mean) in a p-variate normal distribution with covariance matrix $\sigma^2 I$, is distributed, essentially, as χ^2 with p degrees of freedom. The maximum likelihood estimate of σ is to be computed on the basis of a truncated sample of n observations in the range $0 \leq r \leq r_0$, e.g., as in a circular target. Setting $\xi_0 = r_0/\sigma$, the maximum likelihood estimate is obtained by solving

$$\sum_1^n r_i^2/nr_0^2 = \frac{1}{\xi_0}\left[\frac{p}{\xi_0} - \frac{\vartheta_p(\xi_0)}{1 - I_p(\xi_0)}\right] = g_p(\xi_0),$$

for ξ_0 and converting to the equivalent σ where $\vartheta_p(\xi_0)$ is the standard χ^2 density at the point ξ_0 for p d.f. and $I_p(\xi_0)$ is the upper tail area of the same distribution.

Table I gives $G_p(\xi)$ for $p = 2$ and $p = 3$ to 5D for $\xi = .0(.1)4(.5)5$.

LEO KATZ

Michigan State University
East Lansing, Michigan

19[K].—A. HUITSON, "A method of assigning confidence limits to linear combinations of variances," *Biometrika*, v. 42, 1955, p. 471–479.

Consider estimation of a linear function of r unknown population variances $\sigma_1, \cdots, \sigma_r^2$. The estimation data consists of the corresponding statistics, $s_1^2, \cdots, s_r^2$. The s_i^2 are mutually independent and s_i^2/σ_i^2 has a χ^2 distribution with f_i degrees of freedom ($i = 1, \cdots, r$). The problem is to obtain approximate confidence limits for $\sum_1^r \lambda_i \sigma_i^2$, where the constants λ_i are arbitrary but specified. These confidence limits are obtained by determining an approximate expression for the statistic $y = y(s_1^2, \cdots, s_r^2, P)$ with the property that $Pr[\sum \lambda_i s_i^2 / \sum \lambda_i \sigma_i^2 < y] = P$. Let ξ denote the $100P$ percentage point of the normal distribution with zero mean and unit variance and $V_{mn} = (\sum \lambda_i^m s_i^{2m} f_i^{-n})/(\sum \lambda_i s_i^2)^m$. Then, to terms of order $1/f_i^3$, $y = h_0 + h_1 + h_2 + h_3$. Here $h_0 = 1$, $h = \xi\sqrt{2V_{21}}$,

$$h_2 = \xi^2\sqrt{2V_{21}} - 4V_{22}/3V_{21}] - 2V_{32}/3V_{21},$$

$$h_3 = \sqrt{2V_{21}}\{\xi[-4V_{32}(3V_{21})^{-1} + 9V_{43}\sqrt{2}\,V_{21})^{-2} - V_{32}(V_{21})^{-1} - 23(V_{32}/3)^2 V_{21})^{-3}] + \xi^3[-8V_{32}(3V_{21})^{-1} + 2V_{21} - 16(V_{32}/3)^2(V_{21})^{-3} + 5V_{43}(\sqrt{2}\,V_{21})^{-2}.$$

For $r = 2$, the λ_i of the same sign, and $f_i \geq 16$, two decimal accuracy appears to be obtainable from the relation $y = h_0 + h_1, + h_2$. Tables 1–4 contain 2D values of y for $r = 2$; $(\lambda_1 s_1^2 + \lambda_2 s_2^2) = 0(.1)1.$; $f_1, f_2 = 16, 36, 144, \infty$; $P = .01, .05, .95, .99$; and $\lambda_2\lambda_2 > 0$.

J. E. WALSH

Lockheed Aircraft Corporation
Burbank, California

20[K].—C. RADHAKRISHNA RAO & I. M. CHAKRAVARTI, "Some small sample tests of significance for a Poisson distribution," *Biometrics*, v. 12, 1956, p. 264–282.

The authors investigate 3 problems associated with the Poisson distribution particularly if the parameter μ is small: (1) homogeneity of observations, (2) goodness of fit, and (3) deviation in the frequency of any class from its theoretical value, usually the zero class. For testing of homogeneity the authors compare the classical χ^2 test with a newly derived likelihood ratio test. They conclude that the χ^2 test is adequate for $T/f > 3$ where T is the total of the f observations. Four tables are provided for exact tests and in all tables $T = 3(1)10, f = 3(1)10(10)100$. In the first table the critical values of $\sum x_i^2$, x_i being the value of the i-th observation, are given to 1 or 2 S such that the significance level is less than or equal to .05. The actual significance level is tabled. Similarly, the critical values of $\sum x_i \log_e x_i$, the likelihood test for homogeneity, are given to 2 or 3 S. The third table lists the critical values of $\sum f_r \log_e(f_r r!)$, f_r = observed frequency of the variate with value r. This is the likelihood ratio test for the observed distribution to be Poisson. In the fourth table, the authors give the critical value of the frequency of the zero class in the Poisson. The authors also investigate truncated Poisson distributions and illustrate the theory with interesting examples.

L. A. AROIAN

Systems Development Laboratories
Hughes Aircraft Company
Culver City, California

21[K].—HENRY J. HORN, "Simplified LD_{50} (or ED_{50}) calculations," *Biometrics*, v. 12, 1956, p. 311–322.

Let $F(x)$ be a cumulative distribution function and $x_1 < \cdots < x_n$, and let $p_i = r_i/n_i$ be a binomial estimate of $F(x_i)$ based on n_i independent trials. Thompson [1] has given a moving average method for estimating the value of m such that $F(m) = \frac{1}{2}$, i.e., the LD_{50} or ED_{50}, in dose-response assays. In this method only the k values of P_i most nearly bracketing $\frac{1}{2}$ are used. Weil [2] gives tables to facilitate the calculation of m and approximate confidence limits as a function of $r_{k+1}, \cdots, r_{k+k}$ in the special case $n_i = n$ and $\log x_{k+1} = \log x_k + d$. If $k = 4$, $n = 4$ or 5, and the values of x are in geometric progression with $x = 10^{s/2}$ or $x = 10^{s/3}$ (s an integer), the present tables give directly the estimate of m and approximate confidence limits, both to $3S$, for the more probable of the n^k possible outcomes, $(r_{k+1}, \cdots, r_{k+k})$.

PAUL MEIER

The Johns Hopkins University
Baltimore, Maryland

1. W. R. THOMPSON, "Use of moving averages and interpolation to estimate median effective dose," *Bact. Rev.*, v. 11, 1947, p. 115–145.
2. CARROL S. WEIL, "Tables for convenient calculation of median-effective dose (LD_{50} or ED_{50}) and instructions in their use," *Biometrics*, v. 8, 1952, p. 249–263.

22[K].—GOTTFRIED E. NOETHER, "Two sequential tests against trend," Amer. Stat. Assn., *Jn*, v. 51, 1956, p. 440–450.

Table I aids in applying the Wald sequential probability ratio test to a binomial population when H_0: $p = \frac{1}{2}$ and H_1: $p = \frac{1}{2} + \epsilon$. The table is for $\alpha = \beta = .05$ and $\epsilon = .01(.01).49$. Values of the slopes, intercepts, and average sample numbers are given.

Table II contains values of

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_0^{x\sqrt{n}} e^{-\frac{1}{2}t^2} dt$$

for $x = .01(.01)1.00(.02)2.00(.04)3.00$ to 5D.

I. R. SAVAGE

Center for Advanced Study
in the Behavioral Sciences
Stanford, California

23[K].—DOUGLAS G. CHAPMAN, "Population estimation based on change of composition caused by a selective removal," *Biometrika*, v. 42, 1955, p. 279–290.

A population consists of two classes with sizes X_i and Y_i ($X_i + Y_i = N_i$) at times t_i ($i = 0, 1$). Samples of n_0 and n_1 are taken from this population at t_0 and t_1, respectively. Between these two periods, $R_x = X_1 - X_0$ are removed from the X-class and $R_y = Y_1 - Y_0$ from the Y-class. Formulas are given to estimate N_0 and X_0 and the variances of the estimates. Table 1 presents values to 2D of the optimum sample ratio n_1/n_0 to estimate N_0 when $R_y = 0$ for selected values of (R_x/X_0): .01, .02, .05, and .1(.1).5. Table 2 presents simliar values to 3D to estimate X_0 for selected values of (X_0/N_0): .1(.1).9; and (R_x/N_0): .01, .02, and .05(.05).25.

The author also compares the cost of the above method relative to the familiar tagging or capture-recapture procedure. Table 3 presents minimum values to 2D of the relative cost of tagging to the cost of classification such that the variance of N_0 is greater for the tagging procedure; this table is given for the selected values (X_0/N_0) and (R_x/N_0) used in Table 2.

R. L. ANDERSON

North Carolina State College
Raleigh, North Carolina

24[K].—ROLF BARGMANN, "Signifikanzuntersuchungen der einfachen struktur in der Faktoren-analyse," *Mitteilungsblatt für Math. Stat.*, v. 7, 1955, p. 1–24.

Let $P_k = I_{.10}((k-1)/2, \frac{1}{2})$,

$$P(k, r, n) = \sum_{i=r}^{n} \binom{n-k+1}{i-k+1} P_k^{i-k+1}(1-P_k)^{n-i} = I_{P_k}(r-k+1, n-r+1),$$

where $I_x(a, b)$ is the incomplete Beta function. Table I gives the values of P_k to 4D for $k = 2(1)16$. ($P_5 = .1495$ and not the .1459 entered in the table, $P_{12} = .2549$ instead of .2548). The principal table gives integral values of r approximately

satisfying $P(k, r, n) = \alpha$ for $\alpha = .001, .01, .05, .10, .25, .50$; $k = 2(1)12$. The tabled r is usually the largest integer satisfying $P(k, r, n) \leq \alpha$. In some cases the author uses the next lower value of r when the resulting $P(k, r, n) - \alpha$ is small; in these cases the value of r is printed in italics. The rules followed by the author in deciding whether $P(k, r, n) - \alpha$ is small are not clear.

INGRAM OLKIN

Michigan State University
East Lansing, Michigan

25[K].—S. B. CHAUDHURI, "Statistical tables and certain recurrence relations connected with p-statistics," Calcutta Stat. Assn., *Bull.*, v. 6, 1956, p. 181–188.

In samples of n_1 and n_2 from normal p-variate universes, in order to test the hypothesis that the two universes have the same set of variances and covariances the fundamental test statistics are the p roots, $K_1^2, K_2^2, \cdots, K_p^2$, called p-statistics, of the determinantal equation $|A'_{ij} - K^2 A''_{ij}| = 0$ in which (A'_{ij}) and (A''_{ij}) are the variance-covariance matrices of the two samples. With $\lambda = (n_1 - 1)/(n_2 - 1)$ and $\vartheta = \dfrac{\lambda K^2}{1 + \lambda K^2}$ (the maximum ϑ so defined), the authors gives to 2D the 5% and 1% points of ϑ for $p = 2$ and $m = n = 2.5(.5)5(1)11$ where $m = (n_1 - p - 2)/2$ and $n = (n_2 - p - 2)/2$. This extends the table of Nanda [1]. In addition for $p = 3$, 5% and 1% points are here given for ϑ to 2D for $m, n = 0(.5)2$ and $m = n = 2.5(.5)(1)8$.

C. C. C.

1. D. N. NANDA, "Probability distribution tables of the larger root of a determinantal equation with two roots," Indian Soc. Agr. Stat., *Jn.*, v. 3, 1951, p. 175–177. [RMT 1082, *MTAC*, v. 7, 1953, p. 95.]

26[K].—N. MARAKATHAVALLI, "Unbiased test for a specified value of the parameter in the non-central F distribution," *Sankhyá*, v. 15, 1955, p. 321–330.

Given a sample of n observations $\{x_i\}$ from normal populations with common variance σ^2 but unequal means $\{a_i\}$. Then $\sum_{i=1}^{n} x_i^2/\sigma^2$ is distributed as a non-central χ'^2 with n degrees of freedom and non-centrality parameter, $\lambda = \sum_{i=1}^{n} a_i^2/\sigma_2$. The author considers the non-central F'-distribution, which is the ratio of a non-central $\chi_1'^2$ and a central χ^2 with respective degrees of freedom, ν_1 and ν_2, under the null hypothesis, H_0: $\lambda = \lambda_0$. A table is presented to determine the rejection region for an unbiased test of H_0 against the two-sided alternative $\lambda \neq \lambda_0$. The region is: $F' \leq a_1$ or $\geq a_2$, where Prob $(F' \leq a_1 \mid H_0) = \alpha_1$ and Prob $(F' \geq a_2 \mid H_0) = \alpha_2$, $\alpha_1 + \alpha_2 = \alpha = .05$. This table presents values of α_1 and α_2 to 3D, a_1 generally to 2D and a_2 to 1D, for $\nu_1 = 1(1)8$, $\nu_2 = 4(4)20, 40, 60, \infty$ and

$$\varphi = \sqrt{\frac{\lambda_0}{\nu_1 + 1}} = 1, 2.$$

The author also indicates how this table can be used for analysis of variance, correlation ratio, multiple correlation, and discriminant analysis problems.

R. L. ANDERSON

North Carolina State College
Raleigh, North Carolina

27[K].—K. J. RAMACHANDRAN, "On the simultaneous analysis of variance test," *Ann. Math. Stat.*, v. 27, 1956, p. 521–528.

The author first considers the general problem of determining the a_i in the equation

$$P\left[\frac{S_i}{S} \leq a_i t_i/m; i = 1, 2, \cdots, k\right] = 1 - \alpha,$$

where the S_i and S are mutually independent; S_i/σ^2 is distributed as χ^2 with t_i d.f. and S^2/σ^2 as χ^2 with m d.f. A method of evaluating P is presented and then a special method is developed for the case $t_i = t$, for all i. In the latter case P becomes

$$P\left[u = \frac{S_{max}}{S} \leq b\right],$$

where S_{max} is the maximum of the S_i and $b = at/m$ ($a_i = a$, for all i). Upper 5% points of u are given to 2D for $k = 2; t = 1(1)4(2)12, 16, 20$; and

$$m = 5(1)8(2)12(4)24, \infty.$$

R. L. ANDERSON

North Carolina State College
Raleigh, N. C.

28[K].—MARTIN FOX, "Charts of the power of the F-test," *Ann. Math. Stat.*, v. 27, 1956, p. 484–497.

Write f_1 and f_2 for the degrees of freedom of numerator and denominator, respectively, of F. Let α be the size and β the power of F.
Write

$$\phi = [S_e^*/(f_1 + 1)\sigma^2]^{\frac{1}{2}},$$

where S_e^* is the value of S_h^2 when the observable random variables are replaced by their expectations under the alternative hypothesis, and S_h^2 is the sum of squares in the numerator of F. Eight charts show constant ϕ for all combinations of $\beta = .5, .7, .8, .9$, and $\alpha = .01, .05$. The charts are designed to show readily the combination of f_1 and f_2 required to obtain power β against a given alternative. A detailed numerical example, two nomograms facilitating interpolation in β for both values of α, and references to earlier tables are included.

H. A. FREEMAN

Massachusetts Institute of Technology
Cambridge, Mass.

29[K].—H. A. DAVID, "On the application to statistics of an elementary theorem in probability," *Biometrika*, v. 43, 1956, p. 85–91.

The author is concerned with statistical applications of the following theorem in probability. Let $P_{ij\ldots k}$ denote the joint probability of $r(\leq n)$ events A_i, $A_j, \cdots, A_k$; and S_r the sum of the $\binom{n}{r}$ P's with r different subscripts; then the

probability $p_{m,n}$ of the realization of at least m events out of n is given by

$$p_{m,n} = \sum_{l=0}^{n-m} (-1)^l \binom{m+l-1}{m-1} S_{m+l}.$$

This result is applied to the distribution of the extreme deviate from the sample mean, and the maximum F-ratio $s^2_{\max}/s^2$. The author's table 1 gives the value of $P(\lambda)$, the probability to 3D of rejecting the largest of n observations by the use of $x_{\max} - \bar{x}$ at the 5 per cent level of significance, when all the observations are normal with unit variance, $n - 1$ have mean μ and one has mean $\mu + \lambda$, for $\lambda = 1(1)4$, $n = 3(1)10, 12, 15, 20, 25$.

In table 2 are given the probabilities to 4D for various designs of size $l \times l$, (randomized blocks, Latin squares, and Graeco-Latin squares) with which the ordered F-ratio $F_{(t)}$ exceeds the upper percentage point F_α of the corresponding random F-ratio where $l = 3, t = 1, 2, 3; l = 5, 7, 9, \infty; t = 1(1)4, \alpha = .05$, and 0.01.

In table 3 are given the upper 100α percentage points, to $3S$ usually, of the maximum F-ratio followed by the corresponding $100\alpha/n$ percentage points of F for the n-factor designs of table 2 where $l = 3, 5, 6(1)9, n = 2, 3, 4, \alpha = 0.05$ and 0.01.

L. A. AROIAN

Systems Development Laboratories
Hughes Aircraft Company
Culver City, California

30[K].—K. C. S. PILLAI, "On the distribution of the largest or the smallest root of a matrix in multivariate analysis," *Biometrika*, v. 43, 1956, p. 122–127.

The joint distribution of the roots of a certain matrix is well known,

$$p(\vartheta_1, \cdots, \vartheta_s) = C(s, m, n) \prod_1^s \vartheta_i^m (1 - \vartheta_i)^n \prod_{i>j} (\vartheta_i - \vartheta_j),$$

$$0 < \vartheta_1 \leq \cdots \leq \vartheta_s < 1.$$

The author gives approximations for the upper percentage points for the largest (smallest) root for $s = 2, 3, 4, 5$. For $s = 2$, Nanda [1] tabulated the inverse function p^{-1} (.95), P^{-1} (.99) for $m = 0(\frac{1}{2})2, n = \frac{1}{2}(\frac{1}{2})10$, where $P(x) = P(\vartheta \leq x)$. The author tabulates P^{-1} (.95), P^{-1} (.99) to $3S$ for $m = 0(1)4, n = 5(5)30$, $40(20)100, 130, 160, 200, 300, 500, 1000$ using the approximation

$$1 - P(x) = x^{m+1}(1 - x)^{n+1}[2B(m + 1, n + 1)B(2m + 2, 2n + 2)]^{-1}.$$

INGRAM OLKIN

Michigan State University
East Lansing, Michigan

1. D. N. NANDA, "Probability distribution tables of the larger root of a determinantal equation with two roots", Indian Soc. Agr. Stat., *Jn.*, v. 3, 1951, p. 175–177. [RMT 1082, *MTAC*, v. 7, 1953, p. 95.]

31[K].—D. E. BARTON & F. N. DAVID, "Tests for randomness of points on a line," *Biometrika*, v. 43, 1956, p. 104–112.

The present paper is one of a set of papers by the same authors dealing with ordered intervals [1]. On a line of unit length, $(n-1)$ points are dropped randomly forming n intervals. Let the distances from one end of the line be x_i $(i = 1, 2, \cdots, n-1)$ with $x_1 \leq x_2 \leq \cdots \leq x_{n-1}$. Let $d_1 = x_1$,

$$d_i = x_i - x_{i-1} (1 < i < n-1), \qquad d_n = 1 - x_{n-1}.$$

Let $\{g_i\}$ be $\{d_i\}$ with g_1 the smallest d_i, g_2 the next larger, etc. Let r_i be the solution of $d_{r_i} = g_i$. Let r_i' be the solution of $d_i = g_{r_i'}$. The paper deals with distributions of (i)

$$R = \sum_{i=1}^{n} i r_i', \qquad \text{(ii)}\ Y = \sum_{i=1}^{n} i d_i, \qquad \text{(iii)}\ G = \sum_{i=s}^{n} r_i g_i,$$

$$\text{(iii')}\ G_{si} = \sum_{i=s}^{t} r_i g_i, \quad \text{and} \quad \text{(iv)}\ G^* = n \sum_{i=1}^{n} d_i g_i.$$

R and Y are disposed of quickly by references to papers by other authors and by references to the use of the normal approximations to R and to $n - Y$ for moderate or large n. Moments of G_{si} are discussed. Four moments of G are displayed. Table 3 gives μ_1'/n to 4D, μ_2/n to 5D, $n\beta_1$ to 3D and $n(\beta_2 - 3)$ to 3D for $n = 5(5)25, \infty$; $p = s/n = 1/5((1/5)4/5$. μ_1' is the 1st moment, $\beta_1 = \mu_3^2/\mu_2^3$, $\beta_2 = \mu_4/\mu_2^2$, μ_2, μ_3, μ_4 are central moments. For G^*, μ_1', μ_2, β_1 and β_2 are displayed as functions of n and values of β_1 and β_2 are given to 2D in Table 5 for $n = 2(1)10, 20, 40, 100, 400, 1000, \infty$. Percentage points of G^* to 3D and of standardized G^* to 2D are given in Table 6 for $n = 40, 100, 400, 1000$, and for percentiles 0.5, 1, 2.5, 5, 95, 97.5, 99, 99.5.

K. J. ARNOLD

Michigan State University
East Lansing, Michigan

1. D. E. BARTON & F. N. DAVID, "Some notes on ordered random intervals," Roy. Stat. Soc., *Jn.*, ser. B, v. 18, 1956, p. 79–94. (*MTAC*, this issue, Rev. 36, p. 80.)

32[K].—W. D. RAY, "Sequential analysis applied to certain experimental designs in the analysis of variance," *Biometrika*, v. 43, 1956, p. 388–403.

Given the one-way classification model

$$x_{ti} = a + b_t + z_{ti} (t = 1, \cdots, k; i = 1, \cdots, n),$$

where the z_{ti} are uncorrelated with zero means and equal variances, σ^2, the b_t are treatment parameters $(\sum b_t = 0)$. We consider the non-centrality parameter $\delta = \sum b_t/k\sigma^2$ and the testing statistic

$$G = \frac{\text{Treatment Sum of Squares}}{\text{Error Sum of Squares}}.$$

Two hypotheses are considered: $H_0(\delta = 0)$ and H_1 (δ some specified value $\neq 0$). A sequential sampling procedure is set up (consider successive values of n) so that after n samples per treatment one of these 3 decisions is made:

Result of n samples per treatment		*Decision*
$Me^{-\lambda/2} \geq$	$(1 - \beta)/\alpha$	accept H_1
$Me^{-\lambda/2} \leq$	$\beta/(1 - \alpha)$	accept H_0
$\beta/(1 - \alpha) < Me^{-\lambda/2} <$	$(1 - \beta)/\alpha$	take more samples

M is a confluent hypergeometric function,

$$M\left[\frac{kn - 1}{2}, \frac{k - 1}{2}; G/2(1 + G)\right];$$

$\lambda = kn\delta$; and α and β are the usual Type I and Type II error rates.

Let $M(\bar{G})e^{-\lambda/2} = (1 - \beta)/\alpha$ and $M(\underline{G})e^{-\lambda/2} = \beta/(1 - \alpha)$. Values of $\underline{G}$ and $\bar{G}$ to 3D have been computed for $\alpha = \beta = .05$; selected values of n; for $\delta = 0.5$, $k = 2(1)7$; for $\delta = 1.0$, $k = 2(1)10$; for $\delta = 2.0$, $k = 2(1)6$. Certain of the values of $\underline{G}$ and $\bar{G}$ were computed by approximation methods. Tables 7 and 8 present some comparisons of exact and approximate results.

The results are extended to a two-way classification model with k blocks and n treatments (note the unfortunate interchange of n and k from the one-way model). In this case $kn - 1$ is changed to $k(n - 1)$ and $k - 1$ to $n - 1$ in M. Tables of $\underline{G}$ and $\bar{G}$ generally to 3D are presented as before, except that for $\delta = 0.5$ and 1.0, $k = 2(1)8$. Conjectural expected sample sizes are presented to go with these tables, based on a generalization of Wald's formula.

R. L. ANDERSON

North Carolina State College
Raleigh, N. C.

33[K].—S. R. BROADBENT, "Lognormal approximation to products and quotients," *Biometrika*, v. 43, 1956, p. 404–417.

Table 1 of this paper gives upper and lower 1% and 5% points of a variable whose logarithm obeys the normal distribution law in terms of its coefficient of variation, v, to 4D for $v = 0(.0001).15$. The actual percentage point is obtained by multiplying the table entry by the mean value. First differences are given.

C. C. C.

34[K].—D. J. BARTHOLOMEW, "A sequential test of randomness for events occurring in time or space," *Biometrika*, v. 43, 1956, p. 64–78.

The paper is concerned with a sequential test of $a = 0$ against $a = a_0 \neq 0$ for the class of processes for which $Pr\{\text{event in } (T, T + dT)\} = \mu(\mu T)^a dT + 0(dT)$, $(a > -1)$. At the cost of one observation, a transformation eliminates the nuisance parameter μ. The Wald formulas for the operating characteristic and expected sample numbers are then applied. Table 3 gives $E_a(n \mid a_0)$ for various values of

a_0 and a, and for $(\alpha, \beta) = 0.01, 0.05$ where

$$(1) \qquad E_a(n \mid a_0) = \frac{L(a) \log B + [1 - L(a)] \log A}{\log (1 + a_0) - a_0/(1 + a)},$$

$A = (1 - \beta)/\alpha$, $B = \beta/(1 - \alpha)$, $L(a) = (A^{h(a)} - 1)/(A^{h(a)} - B^{h(a)})$, $h(a)$ is the non-zero solution of $1 + ha_0/(1 + a) = (1 + a_0)^h$. The value a' of a for which (1) is indeterminate is near that which maximizes $E_a(n \mid a_0) \cdot E_{a'}(n \mid a_0) = -\log A \log B/[\log (1 + a_0)]^2$. Table 4 gives $E_a(n \mid a_0)$ and $E_{a_0}(n \mid a_0)$ for various values of a_0 and for $(\alpha, \beta) = 0.01, 0.025, 0.05$. Integral values are given in each case. Table 5 lists percentage savings in sample size in using the sequential test rather than a corresponding fixed sample size test.

K. J. ARNOLD

Michigan State University
East Lansing, Michigan

35[K].—H. A. DAVID, "A note on moving ranges," *Biometrika*, v. 42, 1955, p. 512–515.

This publication deals with some of the statistical properties of moving ranges, moving maxima, and moving minima which are of course based on overlapping data. In particular, an expression is obtained for the correlation between two ranges in overlapping samples in terms of means, variances, and covariances of order statistics. With the derived result and the hypothesis of a normal population, the efficiency of mean moving ranges is studied.

Letting $x_{(1)}, x_{(2)}, \cdots, x_{(N)}$ represent the observations ordered in time, we may say that $x_{(i)}, x_{(i+1)}, \cdots, x_{(i+n-1)}$ is the i-th moving sample of size n and designate $m_i(n)$, $m_i'(n)$, and $w_i(n)$ as the corresponding maximum, minimum, and sample or group moving range respectively. Writing the order statistics of a sample of n as $x_1 \geq x_2 \geq x_3 \cdots \geq x_n$, values are given to 4D by David for the correlation between $m_1(n)$ and $m_{1+d}(n)$ for $n = 2(1)5$ and $d \leq (n - 1)$ and for the correlation between two ranges w_1 and $w_1 + d$ in overlapping samples in terms of known moments and cross-product expected values of the x_i. Now the first mean moving range is defined as

$$\omega_i = \sum_{i=1}^{N-n+1} \frac{\omega_i}{N - n + 1}$$

and a mean moving range ω_r can be constructed on every r-th sample. David then studies the efficiency of ω_r, i.e.,

$$\text{Eff} = \text{var } S'/\text{var } \omega_r$$

where S' is the usual (total sample) estimate of population standard deviation

$$S' = \frac{\Gamma(N - 1/2)}{\sqrt{2}\Gamma(N/2)} \sqrt{\sum_{i=1}^{N} (x_i - \bar{x})^2},$$

which is tabulated to 3D for $N = 10(10)60, 50, 100, 200, \infty$, and for $n = 2(1)5$ for ω_1; for $n = 4$ for ω_2; for $n = 5$ for $\bar{\omega}$, the ordinary mean range estimation; and ω^* the most efficient (non-overlapping) weighted mean range estimator [1].

In the absence of trends, $\omega_1(5)$ is found to be considerably more efficient than $\omega_1(2)$, the ordinary mean successive difference. The statistic $\omega_1(2)$ is less influenced by trends in observed data, however, than the statistics $\omega_1(n)$ based on larger n. Compared with $\bar{\omega}$ for groups of 5 observations each and ω^*, the moving range estimators, ω, are seen to gain in efficiency as the total sample size N increases. The relatively low efficiency of the mean moving range for small total sample size N and small group size n is due no doubt to the lack of weighting.

F. E. GRUBBS

Ballistic Research Laboratories
Aberdeen, Maryland

1. FRANK E. GRUBBS & CHALMERS L. WEAVER, "The best unbiased estimate of population standard deviation based on group ranges," Amer. Stat. Assn., *Jn.*, v. 42, 1947, p. 224–241.

36[K].—D. E. BARTON & F. H. DAVID, "Some notes on ordered random intervals," Roy. Stat. Soc., *Jn.*, ser. B, v. 18, 1956, p. 79–94.

In this paper, which is a further investigation of the distribution of the ordered intervals obtained by dropping $n - 1$ points at random on a line segment of unit length, some short tables are included. First the values of the skewness (third standard moment) and the kurtosis (fourth standard moment minus three) are given to 4D for the distribution of the r-th greatest interval for $n = 5(5)25$ and $r = 1(1)n$. Next the coefficient of correlation is given between the two shortest intervals and between the first and third shortest intervals to 3D for $n = 2(1)10$, 15, 25, 50, 100, ∞. Then the limiting values of $\sqrt{n}$ times the skewness, of n times the kurtosis, and of n^2 times the variance divided by the expected value are given to 4 and 5S for the deciles. Turning to the distribution of ratios of two intervals, the 1%, 5%, 10%, 90%, 95%, and 99% points are given for the ratio of the smallest to the largest interval to 3S for $n = 3(1)10$, 15, 20. Finally upper 5%, 2.5%, 1%, and 0.5% points for the interquartile differences of interval length are given 4 or 5D for $n = 7$, 11, 15, 19, 23.

C. C. C.

37[K].—A. E. SARHAN, "Estimation of the mean and standard deviation by order statistics, Part III," *Ann. Math. Stat.*, v. 26, 1955, p. 576–592.

The data considered consists of singly and doubly censored samples from specified types of probability density functions. The best linear estimates of the population mean and standard deviation using sample order statistics, also their variances and their efficiencies (compared to best estimates using uncensored samples), are derived for rectangular and exponential distributions and arbitrary sample size n. The same results are given for sample sizes up to 5 for other types of distributions. The symmetric distributions considered are U-shaped, rectangular, parabolic, triangular, normal, and double (tailed) exponential. Let the smallest r_1 and the largest r_2 sample elements be missing. Table I contains coefficients of the best linear estimates of the mean and standard deviation for these symmetric distributions in the singly censored case. Table II contains the variances and efficiencies for the estimates of Table I. For Tables I and II, $r_1 = 0$, $r_2 = 1(1)n - 2$, $n = 3(1)5$. Tables III and IV contain results of the same nature as Tables I and II,

respectively, for the case of doubly censored samples. In Tables III and IV, $r_1 = 1$, $r_2 = 1(1)n - 1$, and $n = 3, 4$. Results are also computed for a third degree polynomial skewed distribution. Table V contains coefficients of the best linear estimates of the mean and standard deviation of this skewed distribution for singly and doubly censored samples. Table VI contains the variances and efficiencies of the estimates of Table V. Table VII contains a comparison of the variances of the estimates for the skewed distribution with the variances of the corresponding estimates for the exponential distribution. For Tables V, VI, and VII, $n = 3(1)5$ and various combinations of $r_1, r_2\, n - 2$ are considered. With the exception of efficiencies, the results in all Tables are expressed to 7D. Efficiencies are expressed as percentages and to 2D.

J. E. WALSH

Lockheed Aircraft Corporation
Burbank, California

38[K].—V. N. MURTY, "The distribution of the quotient of maximum values in samples from a rectangular distribution," Amer. Stat. Assn., *Jn.*, v. 50, 1955, p. 1136–1141.

These tables are designed to test the hypothesis that two samples have been drawn from the same rectangular population with a known lower bound. Let L_1 be the maximum in a sample of size m, and L_2 the maximum in a sample of size n. The distribution of $u = L_1/L_2$ is obtained, and is independent of the parameter. The power of the test is calculated and 5 percent points of u for samples of size $m, n = 2(1)10$ to 3D is computed.

J. R. VATNSDAL

State College of Washington
Pullman, Washington

39[K].—ALBERT ROSS ECKLER, "Rotation sampling," *Ann. Math. Stat.*, v. 26, 1955, p. 664–685.

This paper shows how to find minimum-variance estimates of the mean $\alpha(t_i)$ of a time-dependent population, assuming that one is restricted to the class of linear unbiased estimates. Each minimum-variance estimate is based on a specified sample pattern (a set of sample values drawn from the population at one or more distinct times). Let the random variable X_{ij} denote the value of element j of the population at time t_i. The correlation between X_{ij} and X_{ik} is assumed to be $\rho^{|i-k|}$; the correlation between X_{ij} and X_{ik} is assumed to be zero; the variance of X_{ij} is assumed to be σ^2 independent of time. Iterative methods are developed; the estimate of the population mean $\alpha(t_{i-1})$ is used in determining the population mean $\alpha(t_i)$.

The paper discusses two important methods of sampling: in one-level rotation sampling, the statistician can add to the sample pattern only sample values that have been drawn from the population at the current time; in two-level (and higher-level) rotation sampling, the statistician can add earlier sample values as well as current ones to the pattern. Schematic sample patterns associated with these two methods are illustrated.

Formula 3.2 gives an iterative scheme for finding the minimum variance estimate of $\alpha(t_i)$. This formula involves an unknown coefficient A_i. Table 3.1 gives the values of A_i to 4D as a function of ρ for $\rho = 0(.1)1, .75, .85, .95$ for the case where half of the elements in the sample at time t_{i-1} are retained in the sample drawn at time t_i, and each of the remaining elements is replaced by a new element. Formula 4.2 gives an iterative scheme for estimating $\alpha(t_i)$. This formula involves an unknown constant a_i. Table 4.1 gives values of a_i for $i = 1(1)11$ as a function of ρ for the same values of ρ as before.

BENJAMIN EPSTEIN

Wayne State University
Detroit, Michigan

40[K].—P. ARMSEN, "Tables for significance tests of 2×2 contingency tables," *Biometrika*, v. 42, 1955, p. 494–511.

In 2×2 tables it is assumed the frequencies result from a double dichotomy of N individuals according to the properties X and Y. The null hypothesis to be tested is: There is no association between X and Y. As is well known, the whole table—for fixed marginal totals—is determined by one of the entries and the probability of any set of frequencies is given by the exact hypergeometric formula. The object of this paper is to provide tables of 5 and 1 per cent significance levels for both one- and two-tailed tests containing up to $N = 50$ observations.

Two possible procedures are given, called D_2: Arrange the possible events in ascending order of the size of their probabilities under the null hypothesis, include in the 100 α per cent two-tailed rejection region whose events for which the cumulative sum of these ordered probabilities is smaller than or equal to α. D_3: Define $F(E)$ or the "first-tail" probability as the cumulative sum of the probabilities under the null hypothesis of all possible events which are more extreme, in the same direction, than a given event E, including E itself. Define $S(E)$ or the "second-tail" probability as the cumulative sum of probabilities, starting with that of the event most extreme in the opposite direction as compared with E, and cumulating up to but not exceeding the value of $F(E)$. If, and only if, $F(E) + S(E) \leq \alpha$, include E in the rejection region for the two-tailed 100 α per cent level of significance.

Tables are computed for both these criteria, there being no difference in most cases. For every possible set of frequencies the tables give significance levels for one- and two-tailed tests for $\alpha = .01$ and $.05$. Explicit directions are given for the use of the tables and examples worked.

J. R. VATNSDAL

State College of Washington
Pullman, Washington

41[K].—A. R. KAMAT, "A two-sample distribution-free test," *Biometrika*, v. 43, 1956, pp. 377–387.

The data consists of two independent samples of sizes m and n with $m \geq n$. These sample values are unequal. The null hypothesis asserts that these samples are from the same population. The alternative hypothesis of interest is that the samples are from populations with different dispersions. The samples are pooled and their values ranked. Let R_n be the range of the ranks for the sample of n and R_m this

range for the sample of m. The test statistic is $D_{n,m} = R_n - R_m - m$. Table 1 contains the largest integer values D_α such that $P(D_{n,m} \leq D_\alpha) \leq \alpha$ and the smallest integer values D_α such that $P(D_{n,m} \geq D_\alpha) \leq \alpha$, for $100\alpha = .5, 1, 2.5, 5$ and $m + n = 7(1)20$ with $n \leq (m + n)/2$. Table 2 contains 3D values of the mean (4D), standard deviation, μ_2, $\sqrt{\beta_1}$, and β_2 of $D_{n,m}$ for $m + n = 16$ and $n = 2(1)8$. Table 3 contains 3D values of the standard deviation, μ_2, β_2 of $D_{n,m}$ for $n = m = 2(1)5, 7, 10, 15, \infty$. Table 4 contains moments and standardized deviates for the limiting distribution of $d = D_{n,m} - m$ for $m/(m + n) = p = .5(.1).9$. The contents of Table 4 are useful in approximating percentage points of $D_{n,m}$ when $m + n > 20$. Table 5 contains a comparison of the approximate and true 2.5 percent points when $m + n = 20$, $p = .5(.1).9$, $m = 10(2)18$. Table 6 contains corrected and standardized upper and lower 2.5 percent points for d to 2D when $\mu = .5(.05).75$.

J. E. WALSH

Military Operations Research Division
Lockheed Aircraft Corporation
Burbank, California

42[K].—J. B. DOUGLAS, "Fitting the Neyman type A (two parameter) contagious distribution," *Biometrics*, v. 11, 1955, p. 149–173.

While moment estimates of the parameters for a two-parameter Neyman Type A distribution are easily computed, the maximum likelihood estimates are obtained by solving a pair of equations known to require a very laborious iteration procedure. The author rewrites these likelihood equations in a form which, with the aid of tables of auxiliary functions, makes it possible to obtain the solution by iterations rather quickly. The case of a two-parameter Neyman Type A distribution truncated by omitting the zero class is also treated. Numerical examples are presented for the complete as well as the truncated distribution. The auxiliary functions p_x and q_x are defined by writing

$$\mu_x' = e^{-\lambda} \sum_{r=0}^{\infty} r_x \lambda r / r!$$

and setting $p_x = \mu_{x+1}'/\mu_x'$; $q_x = p_x(p_{x+1} - p_x)$. Tables of p_x and q_x are given to at least 3 and at most 5S for $x = 0(1)19$, $\lambda = .000(.001).03(.01).3(.1)3$.

Z. W. BIRNBAUM

University of Washington
Seattle, Washington

43[K].—J. H. BENNETT, "Partitions in more than one dimension," Roy. Stat. Soc., *Jn.*, ser. B, v. 18, 1956, p. 104–112.

Though there is some discussion of partitions in more than two dimensions, the greater part of this paper is concerned with bipartitions, extending and clarifying the original work by Fisher [1]. The first table gives values of certain auxiliary functions and the second gives the numbers of bipartitions of $n = 1(1)8$ with given marginal partitions.

C. C. C.

1. R. A. FISHER, "A class of enumerations of importance in genetics," Roy. Soc., *Proc.*, London, Ser. B, v. 136, 1950, p. 509–520.

44[K].—ROBERT HOOKE, "Symmetric functions of a two-way array," *Ann. Math. Stat.*, v. 27, 1956, p. 55–79.

Generalized symmetric functions, (g.s.m.'s) of the elements in an $r \times c$ array are defined as averages of restricted monomial symmetric functions; the restrictions apply to location of the elements in rows and columns. Thus, the g.s.m. of degree $(a + b + c)$ with two elements in the same row and one in a distinct row and a distinct column is

$$\begin{bmatrix} a & b & - \\ - & - & c \end{bmatrix} = \frac{1}{r^{(2)}c^{(3)}} \sum{}' x_{ij}^{a} x_{ik}^{b} x_{mn}^{c}$$

where $\sum'$ indicates summation over distinct subscripts.

Bipolykays are defined by a non-commutative multiplication of the polykays, a family of polynomials in symmetric functions defined by Tukey [1] in generalization of Fisher's cumulants. Therefore, the bipolykays are linear combinations, with integer coefficients, of the g.s.m.'s.

Conversion formulas for the representation of bipolykays in terms of g.s.m.'s and the reverse are given in the text for functions of degree 1 and 2; Table 1 gives the coefficients for degree 3 and Table 2 for degree 4. Multiplication formulas for bipolykays of degree 1 by bipolykays of degree up to 3 are given in the text and, in Table 3, for products of two bipolykays both of degree 2. Table 4 gives the variances and covariances of the bipolykays

$$\begin{pmatrix} 1 & 1 \\ - & - \end{pmatrix}, \qquad \begin{pmatrix} 1 & - \\ 1 & - \end{pmatrix} \text{ and } \begin{pmatrix} 2 & - \\ - & - \end{pmatrix}$$

for matrix bisamples selected by taking from an $R \times C$ population matrix, independently, samples of r from the R rows and c from the C columns and forming the indicated submatrices.

LEO KATZ

Michigan State University
East Lansing, Michigan

1. JOAN W. TUKEY, "Some sampling simplified," Amer. Stat. Assn., *Jn.*, v. 45, 1950, p. 501–519.

45[L].—A. D. SMIRNOV, *Tablitsky Funktsii Éĭri i Spetsial'nykh Vyrozhdennykh Gipergeometricheskikh Funktsii dlia Asimptoticheskikh Reshenii Differentsial'nykh Uravnenii Vtorogo Poriadka* (Tables of Airy Functions and of Special Confluent Hypergeometric Functions for Asymptotic Solutions of Differential Equations of the Second Order). Izdatel'stvo Akad. Nauk SSSR (Press of the Academy of Sciences of the USSR), Moscow, 1955. 264 p. + two loose cards, 26.6 cm. Price 28 rubles.

This volume in the series of Mathematical Tables of the Computational Centre of the Academy of Sciences of the USSR contains tables of three types of functions. Apart from their occurrence in various physical investigations, the functions play an important part in work of A. A. Dorodnitsyn on asymptotic solutions for

large λ of second-order linear differential equations

$$[k(x)y']' + [\lambda^2 r(x) + q(x)]y = 0$$

when $r(x)$ and $q(x)$ have singularities (zeros of poles) in an interval. The tables were in fact prepared on the initiative of Dorodnitsyn and in constant consultation with him. Some indications concerning this work on asymptotic solutions are given in the Introduction; for further details, reference is made to Dorodnitsyn's memoir [1].

The functions first considered are solutions of Airy's equation $U''(s) + sU(s) = 0$. The particular solutions $U_1(s, 1)$, $U_2(s, 1)$ which are tabulated, along with their first derivatives $U_1'(s, 1)$, $U_2'(s, 1)$, are such that at $s = 0$ we have $U_1 = U_2' = 1$, $U_2 = U_1' = 0$. Consequently

$$\begin{aligned} U_1(s, 1) &= \sum_{n=0}^{\infty} (-1)^n 1 \cdot 4 \cdot 7 \cdots (3n-2) s^{3n}/(3n)! \\ &= 3^{-1/3}\Gamma(\tfrac{2}{3}) s^{1/2} J_{-1/3}(\tfrac{2}{3}s^{3/2}), \\ U_2(s, 1) &= \sum_{n=0}^{\infty} (-1)^n 2 \cdot 5 \cdot 8 \cdots (3n-1) s^{3n+1}/(3n+1)! \\ &= 3^{1/3}\Gamma(\tfrac{4}{3}) s^{1/2} J_{1/3}(\tfrac{2}{3}s^{3/2}). \end{aligned}$$

The two functions and two derivatives are tabulated, with smoothed second differences, for $s = 0(.01)10$ to 5D on pages 27–46, and for $-s = 0(.01)6$ to 5D or 5s on pages 47–58. Also tabulated on pages 27–46, equally for $s = 0(.01)10$ and to 5D, but without differences, are three functions $V_{11}(s)$, $V_{12}(s)$, $V_{22}(s)$ which are integrals of expressions involving the U functions (explicit formulas are given at the bottom of page 7).

The functions tabulated invite comparison with the $Ai(x)$, $Bi(x)$ solutions of $y'' = xy$ which are employed in the British Association tables [2] and are now in common use. Formulas on page B17 of [2] show that $Ai(x)$ and $Bi(x)$ are linear combinations of two solutions, $y_1(x)$ and $y_2(x)$, analogous to the U solutions; in fact, $U_1(s, 1) = y_1(-s)$, $U_2(s, 1) = -y_2(-s)$. The Ai, Bi functions behave more neatly as $x \to -\infty$, whereas the y (or U) solutions behave more simply near $x = 0$ (or $s = 0$). Since cases do arise in which both boundary conditions relate to the origin, it is useful to have the U functions tabulated as well as the Ai, Bi functions.

The second batch of functions are solutions $U_1(s, \alpha)$, $U_2(s, \alpha)$ of a generalized Airy equation $U''(s) + s^\alpha U(s) = 0$, in which, if complications are to be avoided, $\alpha + 2$ (supposed positive) must not be the reciprocal of an integer. The particular solutions denoted by $U_1(s, \alpha)$, $U_2(s, \alpha)$ are again such that at $s = 0$ we have (for $\alpha > -1$) $U_1 = U_2' = 1$, $U_2 = U_1' = 0$. Power series are given, as also are expressions in terms of Bessel functions of order $\pm 1/(\alpha + 2)$. On pages 61–156 the functions $U_1(s, \alpha)$, $U_2(s, \alpha)$ with first differences, and also the derivatives $U_1'(s, \alpha)$, $U_2'(s, \alpha)$ without differences, are tabulated to 4D for $s = 0(.01)6$ and the following

16 values of α(6 pages for each α):

$$\alpha = \tfrac{1}{4}, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{3}{4}, \tfrac{5}{4}, \tfrac{4}{3}, \tfrac{5}{3}, \tfrac{5}{3}, \tfrac{7}{4}, 2,$$

$$-\tfrac{1}{4}, -\tfrac{1}{3}, -\tfrac{1}{2}, -\tfrac{2}{3}, -\tfrac{3}{4}.$$

The third set of functions are solutions $V_1(s, p_0)$, $V_2(s, p_0)$ of the special confluent hypergeometric equation

$$V''(s) + [\tfrac{1}{2}p_0(1 - \tfrac{1}{2}p_0)s^{-2} + s^{-1}]V(s) = 0$$

in which complications arise when p_0 is integral. The particular solutions denoted by $V_1(s, p_0)$, $V_2(s, p_0)$ are normally the power series beginning with $s^{1/2p_0}$ and $s^{1-1/2p_0}$ respectively; the appropriate modification when $p_0 = 1$ is indicated, since the tabulation extends to this value of p_0. Although the fact does not appear to be stated, the solutions are simply expressible in terms of Bessel functions. When p_0 is not integral, we have

$$V_1(s, p_0) = \Gamma(p_0)s^{1/2}J_{p_0-1}(2s^{1/2}),$$

$$V_2(s, p_0) = \Gamma(2 - p_0)s^{1/2}J_{1-p_0}(2s^{1/2}).$$

When $p_0 = 1$, the first formula remains valid, giving $V_1(s, 1) = s^{1/2}J_0(2s^{1/2})$, while the second is replaced by $V_2(s, 1) = \tfrac{1}{2}\pi s^{1/2}Y_0(2s^{1/2})$, where Y_0 (sometimes called N_0) is the Bessel function of the second kind in the form now usual. On pages 159–258 the two functions $V_1(s, p_0)$, $V_2(s, p_0)$ with first differences, and the first derivatives $V_1'(s, p_0)$, $V_2'(s, p_0)$ without differences, are tabulated to 4D for $s = 0(.01)6$, $p_0 = .1(.1)1$. (In the sub-title of the table, on page 157, $0.1 \leq p_0 \leq 10$ is a misprint for $0.1 \leq p_0 \leq 1.0$.) On page 259 are tabulated to 4D or 5S the first five zeros, s_1 to s_5, of both $V_1(s, p_0)$ and $V_2(s, p_0)$, again for $p_0 = .1(.1)1$.

On page 260 (and also on one loose card) coefficients $\frac{1}{2}u(u + 1)$ of the second difference are tabulated to 4D without differences for $-u = 0(.001)1$. On page 20 (and also on a larger scale on the other loose card) is a nomogram for evaluating $\frac{1}{2}u(u + 1)\Delta_2$. On page 264 are given corrigenda to three other tables; see *MTAC*, v. 11, 1957, p. 125–126.

A. F.

1. A. A. DORODNITSYN, "Asymptotic laws of distribution of eigenvalues for certain forms of differential equations of the second order," *Uspekhi Matem. Nauk* (N.S.), v. 7, 1952, p. 3–96. For Russian title see RMT 1113, *MTAC*, v. 7, 1953, p. 179.

2. J. C. P. MILLER, *The Airy Integral*, Part-volume B of British Assoc. Adv. Sc., Math. Tables, Cambridge University Press, 1946.

46[L].—NUMERICAL COMPUTATION BUREAU, Report No. 8, "Tables of Whittaker Functions," The Tsuneta Yano Memorial Society (1–9 Yuraku-cho, Chiyoda-Ku, Tokyo, Japan), 1954, 26 cm., 39 p. Price $1.00, and

[L].—NUMERICAL COMPUTATION BUREAU, Report No. 9, "Tables of Whittaker Functions (Wave Functions in Coulomb Field)," The Tsuneta Yano Memorial Society (1–9 Yuraku-cho, Chiyoda-Ku, Tokyo, Japan), 1956, 66 p., 26 cm. Price $3.00.

These two reports contain tabulations of the solutions to the differential equation

$$\frac{d^2W}{dz^2} + \left\{-\tfrac{1}{4} + \frac{k}{z} + \frac{\tfrac{1}{4} - m^2}{z^2}\right\}W = 0$$

known as Whittaker functions [1]. One form of the solutions, due to Whittaker, is denoted by $W_{k,m}(z)$ and $W_{k,-m}(z)$; these are sometimes written in the form

$$W_{k,m}(z) = e^{-1/2z}z^k u_{k,m}(z)$$

and the first derivative

$$\frac{dW_{k,m}(z)}{dz} = e^{-1/2z}z^k v_{k,m}(z).$$

Another form of the solutions is commonly denoted by $M_{k,m}(z)$ and $M_{k,-m}(z)$; $M_{k,m}(z)$ is related to the confluent hypergeometric function ${}_1F_1(\alpha, \beta, z)$ by

$$M_{k,m}(z) = z^{1/2+m}e^{-1/2z}\,{}_1F_1(\tfrac{1}{2} + m - k, 2m + 1; z).$$

The tabulation is for $m = \frac{1}{2}$ and k in the range 0 to 1. Recurrence relations for generating solutions for other k and m are given in Report No. 8. The first derivative of the Whittaker functions is tabulated for use in the recurrence relations.

Report No. 8 contains the following tables:

(a) $M_{k,1/2}(z)$, $\frac{d}{dz}M_{k,1/2}(z)$ with least significant figure 10^{-6}, $z = 0.0(0.1)5.0$ and $k = 0.00(0.05)1.00$. Second central differences with z and k as variables are also given.

(b) $u_{k,1/2}(z)$, $v_{k,1/2}(z)$ with least significant figure 10^{-6}, $z = 5.0(0.2)25.0$ and $k = 0.05(0.05)0.95$; $ku_{k,1/2}(z)$, $kv_{k,1/2}(z)$ with least significant figure 10^{-6}, $1/z = 0.1(0.1)0.4$ and $k = 0.05(0.05)0.95$. Second central differences with z and k as variables are also given.

(c) $E_2(\alpha)$, $E_2(\beta)$ with least significant figure 10^{-4}, $\alpha = 1 - \beta = 0.00(0.01)0.50$, where $E_2(\alpha)$ and $E_2(\beta)$ are the interpolation coefficients appearing in Everret's interpolation formula.

(d) Log z with least significant figure 10^{-4}, $z = 1.00(0.01)2.00(1.00)10.00$, 100.

(e) e^{-z} with least significant figure 10^{-4}, $x = 0.00(0.01)1.00$ and with four significant figures $x = 1.00(1)10(10)100$.

Report No. 9 is a table of Coulomb wave functions. It gives

$$\exp\left(-\frac{x}{2}\xi\right)\exp\left[-i\left(\frac{l}{2}\pi - \sigma_l\right)\right]W_{i\xi,1/2}(-ix)$$

where ξ and x are real, $l = m - \frac{1}{2}$, and $G_{\xi,1/2}(x)$ and $F_{\xi,1/2}(x)$ are defined as the real and imaginary parts of this function. As $x \to 0$, $G_{\xi,1/2}(x) \to \infty$ so in the neighborhood of $x = 0$, a function defined by the authors as

$$\Gamma_{\xi,1/2}(x) = G_{\xi,1/2}(x) + \frac{\xi}{2C^2_{\xi,1/2}}\log x F_{\xi,1/2}(x)$$

which remains finite at zero is tabulated rather than $G_{\xi,1/2}(x)$. This formula for $\Gamma_{\xi,1/2}(x)$ appears inconsistent with other equations presented and with the tables

themselves. The correct equation for $\Gamma_{\xi,1/2}(x)$ appears to be

$$\Gamma_{\xi,1/2}(x) = G_{\xi,1/2}(x) + \left(\frac{1 - e^{-2\pi\xi}}{\pi}\right) F_{\xi,1/2}(x) \log x.$$

For $x \gg 1$ the asymptotic form of $W_{i\xi,1/2}(-ix)$ is:

$$W_{i\xi,1/2}(-ix) = u(x) \exp\left\{i\left[\frac{x}{2} + \xi \log x + \sigma_0(\xi) + \phi(x)\right]\right\}$$

and

$$\frac{dW_{i\xi,1/2}(-ix)}{dz} = \frac{d}{dz}\left[G_{\xi,1/2}(x) + iF_{\xi,1/2}(x)\right]$$

$$= \frac{dF_{\xi,1/2}(x)}{dx} - i\frac{dG_{\xi,1/2}(x)}{dx}$$

$$= \tfrac{1}{2}v(x) \exp\left\{i\left[\frac{\pi}{2} + \xi \log x + \sigma_0(\xi) + \chi(x)\right]\right\}$$

where $\sigma_0(\xi) = \arg \Gamma(1 - i\xi)$. The functions u, ϕ, v and χ are tabulated for $x \geq 5$.

Report No. 9 contains the following tables:

(a) $F_{\xi,1/2}(x)$, $\frac{d}{dx} F_{\xi,1/2}(x)$, $\Gamma_{\xi,1/2}(x)$, $\frac{d}{dx} \Gamma_{\xi,1/2}(x)$ with least significant 10^{-6}, $x = 0.00(0.05)1.00$, $\xi = 0.00(0.02)0.50(0.05)1.00$. Second central differences with ξ and x as variables are also given.

(b) $F_{\xi,1/2}(x)$, $\frac{d}{dx} F_{\xi,1/2}(x)$, $G_{\xi,1/2}(x)$, $\frac{d}{dx} G_{\xi,1/2}(x)$ with least significant figure 10^{-6}, $x = 1.00(0.05)2.00(0.10)5.00$ and $\xi = 0.00(0.02)0.50(0.05)1.00$. Second central differences with ξ and x as variables are also given.

(c) $u_{\xi,1/2}(x)$, $\phi_{\xi,1/2}(x)$, $v_{\xi,1/2}(x)$, $\chi_{\xi,1/2}(x)$ with least significant figure 10^{-6}, $1/x = 0.000(0.005)0.200$ and $\xi = 0.1(0.1)1.0$. Second central differences with ξ and x as variables are also given.

These tables are among the most complete tabulations of these functions currently available. They are well laid out and easily read. Unfortunately, however, they suffer from an inadequate explanation of how to use them There is in fact an error in the definition of $\Gamma_{\xi,1/2}(x)$ in Report No. 9 as cited above. The notation is also difficult to follow. Furthermore it should be noted that the functions $F_{\xi,1/2}(x)$ and $G_{\xi,1/2}(x)$ tabulated do not have the same normalization as the corresponding functions cited in the reference [2] given by the authors of the tables. An extensive discussion of the Coulomb wave functions with many formulae may be found in the National Bureau of Standards Tables [3]. No discussion of the method by which the tables were computed or of the estimated error is presented.

LLOYD D. FOSDICK

Digital Computer Laboratory
University of Illinois
Urbana, Illinois

1. EDMUND TAYLOR WHITTAKER & G. N. WATSON, *Modern Analysis*, Cambridge University Press, 1943.
2. T. YAMANOUCHI & M. KOTANI, "Excitation of atoms by electron collision," Phys. Math. Soc., *Proc.*, Japan, v. 22, 1940, p. 14–33.
3. NBS Applied Mathematics Series No. 17, *Tables of Coulomb Wave Functions*, v. 1, U. S. Gov. Printing Office, Washington, D. C., 1952.

47[L].—J. W. HEAD & W. P. WILSON, *Laguerre Functions: Tables and Properties*, Monograph No. 183R, The Inst. of Elec. Engineers, London, 1956, 13 p. + 5 tables + 4 p. of appendices, 27 cm. Copies obtainable from The Institution of Electrical Engineers, Savoy Place, London, W. C. 2. Price 1s.3d.

This paper tabulates the Laguerre functions $e^{-x} L_n(x)$ for $n = 0(1)20$, $x = 0(.1)1(.2)3(.5)6(1)14(2)40(5)100$. Most of the entries are given to four decimals. For large values of the argument two significant figures are given. In addition there are zeros for the same values of n to six decimals. Comparison of these zeros with existing tables [1] for $n = 1(1)16$ serve to substantiate the author's claim that the values are uncertain in the fifth decimal place. However, the discrepancy is never more than one unit. Accompanying the tables is an attractive computation of the most important mathematical properties which should certainly serve the needs of engineers. An omission noted is Tricomi's expansion [2] for $L_n(x)$ in terms of Bessel functions $J_n(2\sqrt{2x})$.

M. ABRAMOWITZ

National Bureau of Standards
Washington, D. C.

1. HERBERT E. SALZER & RUTH ZUCKER, "Tables of the zeros and weight factors of the first fifteen Laguerre polynomials," NBS Applied Mathematics Series No. 37, *Tables of Functions and Zeros of Functions*, 1954, p. 191–199. [Amer. Math. Soc., *Bull.*, v. 55, 1949, p. 1004–1012.] [*MTAC*, Rev. **104**, v. 10, 1956, p. 249–250.]

2. HARRY BATEMAN, *Higher Transcendental Functions*, v. 2 (A. Erdélyi, Director), McGraw-Hill Book Co., Inc., New York, 1953, p. 199. [*MTAC*, Rev. **67**, v. 11, 1957, p. 114–116.]

48[M].—E. J. SCOTT, *Transform Calculus with an Introduction to Complex Variables*, Harper and Brothers, New York, 1955, viii + 330 p., 23 cm. Price $7.50.

The present book is intended for those who wish to learn the application of transform theory to the solution of problems in engineering, physics, and chemistry. It is devoted mostly to the theory and application of the Laplace transform, although in the final chapter the finite Fourier and Hankel transforms are introduced. The bulk of the book consists of the application of the Laplace transform to a great variety of problems in diffusion, vibrations of rods, deflection of beams, electrical transmission, etc. The early portion contains a concise but readable account of the necessary mathematical background.

Chapter I is a summary of the elementary portion of the theory of analytic functions centering around Cauchy's integral theorem: Taylor and Laurent series, singularities, residues, and contour integration. In Chapter II the Laplace transform is defined, its existence established under suitable conditions, and the complex inversion formula obtained. This latter takes for granted and as point of departure the Fourier integral theorem. Chapter III is devoted to formal properties of the transform-translations, convolutions, etc. The following six chapters are devoted to a wealth of applications: ordinary linear differential equations and systems of such, difference equations, partial differential equations, special functions, asymptotic series, and integral equations. The final Chapter X defines and applies the finite Fourier and Hankel transforms. Tables of the several transforms are appended.

Some, but not many, misprints were noted. The integrand of the last integral of equation $8 \cdot 2 \cdot 6$ should have ξ^2 in its denominator instead of 2.

Particularly because of the abundance of well worked out examples, the engi-

neer or physical scientist should find this a very useful book, and it should provide him with a broad background of problems to which the transform method is applicable.

The tables appended to the book consist of

(a) A table of operations with Laplace Transforms (9 entries)
(b) A table of Laplace Transforms (111 entries)
(c) Tables of finite sine transforms (14 entries) and cosine transforms (11 entries)

JOHN W. GREEN

University of California
Los Angeles, California

49[P].—T. Y. CHEN, C. P. SIESS, & N. M. NEWMARK, "Moments in simply supported skew I-beam bridges," University of Illinois Engineering Experiment Station Bulletin No. 439, Studies of Slab and Beam Highway Bridges, Part VI, 1957, 72 p., 28 cm. Price $1.00.

This investigation is concerned with the behavior of simply-supported skew I-beam bridges. This type of structure is used extensively in this country for spans up to about 100 ft. It consists of a reinforced concrete slab in the form of a parallelogram, supported on a series of five uniformly spaced, flexible girders which are parallel to the direction of traffic and are supported on abutments which, in general, are *not* perpendicular to the girders.

Analyses have been made for a number of structures of this type for various proportions and relative stiffnesses of the slab and the beams, and for different angles of skew. From the results of these analyses general relations have been obtained for the rational design of this type of structure.

The mathematical expressions governing this problem are: the differential equation for flexure of slabs,

$$N\nabla^2(\nabla^2 w) = p \tag{1}$$

where

$$N \frac{EI}{1 - \mu^2},$$

a constant related to the properties of the slab,

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},$$

the Laplace operator, p is the intensity of the external load and w is the unknown deflection of the slab at a point defined by the rectangular coordinates x and y; the differential equations for the moments M_x, M_y, M_{xy} are

$$\begin{aligned} M_x &= -N\left(\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2}\right) \\ M_y &= -N\left(\frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2}\right) \\ M_{xy} &= -N(1 - \mu) \frac{\partial^2 w}{\partial_x \partial^y}; \end{aligned} \tag{2}$$

the differential equation for the deflection w of a beam

$$E_b I_b \frac{d^4 w}{dx^4} = q; \tag{3}$$

and the equation for the moment of the beam

$$M_b = -E_b I_b \frac{d^2 w}{dx^2} \tag{4}$$

where $E_b I_b$ is the flexual rigidity of the beam and q is the load per unit length of the beam.

The problem treated in this report is the determination of w which satisfies equations (1) and (3) and which in addition satisfies the boundary conditions of the problem. Once w, the deflection of the structure has been obtained, the bending moments (from which the stresses are evaluated) may be determined from equations (2) and (4).

Because of the nature of the boundary conditions, an exact solution of the governing differential equations does not appear to be possible. In this investigation the problem was analyzed by the method of finite differences using a network of points in skew coordinates, in which the elemental mesh is a parallelogram formed by lines parallel to the sides of the skew panel. A total of 63 node points were considered. By writing a difference equation for each point, there results, in general, a set of 63 simultaneous algebraic equations, the solution of which gives the values of the deflection at the nodal points. In this study, the load on the bridge was broken into symmetrical and antisymmetrical components. In this way the analysis of each particular problem was reduced to the solution of two sets of equations, one set having 32 equations and the other 31. The complete solution was obtained by adding the solutions for the symmetrical and antisymmetrical components of loading. The choice of the number of nodal points considered was governed by the capacity of the ILLIAC which could solve 39 simultaneous equations at the time this investigation was conducted (1954). (At present the ILLIAC can solve as many as 142 simultaneous equations.)

A total of 18 structures having different geometrical and physical characteristics were analyzed in this investigation. These analyses necessitated the solution of 306 sets of simultaneous equations (each having 31 or 32 unknowns). It took the ILLIAC 3.5 minutes to solve each set, or about 17 hours to solve the 306 sets.

The coefficients of these equations were functions of the angle of skew, ϕ, the ratio of the sides, b/a, and the relative stiffness of the slab and the beams, H. As the evaluation of these coefficients on a desk calculator and the preparation of tapes on which numerical values of about 1000 coefficients were to be punched for each set of equations would be most laborious and time-consuming, a special program was prepared which permitted the evaluation of these coefficients within the ILLIAC for specified values of ϕ, b/a, and H.

Contour maps, giving the coefficients for the bending moment at midspan of the second beam due to a single concentrated load applied at any point on the bridge were obtained. For other loads the value of the bending moment is obtained by multiplying the coefficients given on the diagram by Pa, where P is the magnitude

of the concentrated load and a is the span of the bridge. The bending moment due to multiple wheels can be found by superimposing the effect of each load.

From influence surfaces of this type, maximum live load moments were obtained for standard trucks and the results were used to develop general relationships which provide a more rational basis for the design of skew I-beam bridges.

A. H. T.

50[S].—PHILIP M. MORSE & HUSEYIN YILMAZ, *Tables for the Variational Determination of Atomic Wave Functions*, The Technology Press, Massachusetts Institute of Technology, Cambridge, Mass., 1956, 85 p., 30 cm. Price $2.50 plus 15c for mailing.

This set of tables should prove useful in the many instances when one desires an atomic wave function sufficiently accurate to distinguish between multiplets but simple enough to be expressed in closed form for easy use in subsequent calculation. Such functions should be welcomed when a modern high-speed computer is either not available or practical. The wave functions in point are to be constructed as Slater determinants of hydrogen-type orbitals; the energy of the atom is computed and minimized with respect to the parameters contained in the orbitals in order to obtain the values which characterize the most acceptable function. This yields a wave function which neglects inter-election correlation.

The tables embrace ten hydrogen functions: the one $1s$, the one $2s$, the three $2p$, and the five $3d$ orbitals. Each group is characterized by one parameter. Expressions are given for the kinetic energy, potential energy, and exchange energy contributed by any combination of such electrons to the total energy. It is these expressions which are tabulated as function of the parameters so that the minimization can be conveniently carried out on a desk calculator. Even those energies which are trivial functions of the parameters are tabulated. The functions tabulated are expressible in terms of combinations of hypergeometric functions and were computed on IBM CPC equipment. No discussion of numerical methods or procedures is given. All tables are given to at least 5 decimal places. A recipe for carrying out the construction and minimization procedures along with a rather complete example is given. No discussion of the errors to be expected in this type of variational wave function is included nor is a comparison between a worked out function of this type and other better or worse types of function given for some particular case.

JAMES N. SNYDER

Digital Computer Laboratory
University of Illinois
Urbana, Illinois

51[S].—A. J. FERGUSON & A. R. RUTLEDGE, *Coefficients for Triple Angular Correlation Analysis in Nuclear Bombardment Experiments*, CRP-615, Atomic Energy of Canada Limited, 1957, xvi + 469 p. 27 cm. Price $6.00.

Here is an ambitious table, in a somewhat difficult format, for the nuclear bombardment problem of studying nuclear states by measuring the angular correlations

among the directions of an entering particle and two gamma rays emitted in cascade. The tabulated expansion coefficients depend on some thirteen parameters, which explains most of the difficulty in the tabular format. The remaining difficulty is attributable to the suppression of zero coefficients.

The introduction is clear enough for students of the problem. It is quite opaque to others, but adequate references are given. There is a well-marked sample page.

The ranges covered are as follows, where s is the channel spin, the l's are allowed incident orbital momenta, the J's are the spins of the first and second states, I is the spin of the final state, and the L's are the allowed multipolarities of the gamma rays. s: 0(1/2)9/2, with 8/2 omitted. l: 0(1)3. J: 0(1/2)5. I: 0(1/2)7/2. L: 1(1)2.

R. LAZARUS

Los Alamos Scientific Laboratory
Los Alamos, New Mexico

52[S].—A. V. H. MASKET, R. L. MACKLIN, & H. W. SCHMITT, *Tables of Solid Angles and Activations*: I. *Solid Angle Subtended by a Circular Disc*; II. *Solid Angle Subtended by a Cylinder*; III. *Activation of a Cylinder by a Point Source*, ORNL-2170, United States Atomic Energy Commission, Technical Information Service Extension, Oak Ridge, Tennessee, Nov. 1956, 18 + 57 p., 27 cm.

There are three tables, furnishing approximations to the activation integral $I = \int_V dv \exp(-l/\lambda)/R^2$ in the limiting cases of $l/\lambda \gg 1$, ($I \simeq \Omega$, the solid angle) and $l/\lambda \ll 1$ ($I \simeq \int_V dv/R^2$), where V is a right circular cylinder. The application in view is the activation of a cylindrical detector by a point radiation source (or a point detector by a uniform cylindrical source). R is the total distance, and l the distance inside V, from the point source to dv; λ is the mean free path in V. A correction and error estimation procedure is given for l/λ small but not negligible. It is suggested that some correction can be made for elastic scattering of the incident radiation by using certain values in Table III to estimate an effective increased path length. The introduction is reasonably complete; additional mathematical details can be found in [1] [2].

Table I contains the solid angles subtended by a circular disc of unit radius at a point in a plane h units above the disc plane and ρ units from the disc axis. The entries are for h = 0.1(0.1)5.0(0.2)10.0(1.0)20.0(2.0)100, and ρ = 0.0(0.1)0.9(0.05)1.0(0.1)6.2(0.3)6.5(0.5)16(1.0)36.

Table II contains the solid angles subtended by a right circular cylinder of unit radius and height h units at a point, P, in the base plane ρ units from the axis. Solid angles subtended at points in a plane cutting the cylinder can be found by addition of the entries for the upper and lower parts of the cut cylinder; those at points in a plane below the base must be found by the difference of two values, corrected by addition of the base term from Table I. The entries are for the same values, excluding $\rho \leq 1$.

Table III contains the values of the integral $I = \int_V dv/R^2$ over the cylinder of Table II, where R is the distance from P to dv. The entries are for the same values as in Table I, except for the inclusion of ρ = 0.85 and the omission of ρ = 6.2.

All entries are rounded to six significant figures and are of the form 0872425σ03,

meaning $0.872425\ x + 10^{\pm 3}$, + or − for σ printed as O or F, respectively. The character size is unfortunately rather small.

ROGER LAZARUS

Los Alamos Scientific Laboratory
Los Alamos, New Mexico

1. A. V. H. MASKET, R. L. MACKLIN, & H. W. SCHMITT, "Cylinder radiation and activation," *Rev. Sci. Instr.*, v. 28, 1957, p. 189.
2. A. V. H. MASKET, "Solid angle contour integrals, series, and tables," *Rev. Sci. Instr.*, v. 28, 1957, p. 191.

53[W, Z].—C. E. SHANNON & J. MCCARTHY, Editors, *Automata Studies*, Annals of Mathematics Studies No. 34, Princeton University Press, 1956, viii + 286 p., 23 cm. Price $4.00.

This collection of papers is classified under three headings: (1) Finite Automata, (2) Turing Machines, and (3) Synthesis of Automata. The papers by Kleene, Moore, and von Neumann under (1) and those by Shannon, Davis, and De Leeuw, Moore, Shannon, Shapiro under (2) obtain carefully worked out results of mathematical interest. The papers under (3) are concerned with simluating aspects of animal behavior.

S. C. Kleene in "Representation of Events in Nerve Nets and Finite Automata" obtains characterizations of the behavior of automata first postulating past time as infinite and then postulating the contrary. The principal result for automata with infinite past is that for every automaton there is an automaton without cycles which behaves in the same way (this follows from theorems 1, 6 and lemma 1) and this behavior can be described by a (finite) set of tables all of the same finite dimensions. The top row of a table specifies that input state at any time t, the next row the input state at $t - 1$, etc. These tables specify exactly those sequences of input states to which the automaton responds. Conversely, given a set of such tables there exists an automaton whose behavior it represents. This leads to the more significant and more complicated theory of automata with finite past. The notion of "regular event" is defined and it is proved that automata (with finite past) can detect all regular events and only these. The class of k-input regular events (those which k-input automata detect) could be defined as the smallest class of sets of sequences of k-tuples of zeros and ones which includes all the unit sets and contains $\alpha \cup \beta$, $\alpha \cdot \beta$ and α^* whenever the class contains α and β. Here $\alpha \cdot \beta$ is the set of sequences obtained by prolonging any sequence of α with a sequence from β and $\alpha^* = \alpha \cup \alpha \cdot \alpha \cup \alpha \cdot \alpha \cdot \alpha \cup \cdots$. Concrete examples of automata are obtained by interconnecting (models of) neurons. The resulting configuration is called a nerve net.

The notion of indistinguishability is central for E. F. Moore "Gedanken-Experiments on Sequential Machines." With each initial internal state q of an automaton A a transformation $T_A(q)$ may be defined which when applied to a finite sequence of input states has as value the determined sequence of output states. The author's concept of indistinguishability of two states q and q' of automata A and A', respectively, may be formulated as the requirement that $T_A(q) = T_{A'}(q')$. The author's concept of indistinguishability of two automata seems odd. It may be formulated as follows: there exist functions f, g such that $T_A(q)[i] =$

$T_{A'}(f\ (i,q))\ [i]$ and $T_{A'}(q')[i] = T_A(g(i, q'))\ [i]$, where i varies over finite sequences of input states, q over internal states of A and q' over internal states of A'. If A is strongly connected (i.e., for any ordered pair of internal states there exists a sequence of input states which takes the first internal state into the second) then f may be chosen so as not to depend on i. Thus, when applied to strongly connected machines, the notion of indistinguishability becomes natural. The author shows that among all automata indistinguishable from a given strongly connected one there exists an (essentially) unique automaton with a minimal number of internal states. Besides qualitative theorems of the kind described above, the author obtains quantitative theorems typified by the following result. If an automaton A has the property that each two of its internal states are distinguishable then for any pair q, q' of internal states there exists a sequence s of input states of length $n - 1$ such that $T_A\ (q) \neq T_A(q')$, where n is the number of internal states of A.

In "Probabilistic Logics and the Synthesis of Reliable Organisms from Unreliable Components," J. von Neumann introduces a technique for constructing reliable finite automata out of elements whose probability of improper operation is positive. The technique is called multiplexing. Instead of using, say, an element with two input lines and one output line, one uses a network of elements (a generalized element) with two bundles of N (N is usually a large positive integer) input lines and an output bundle of N lines. It is assumed a certain fiduciary level Δ is fixed (the same for all bundles) with $0 < \Delta < \frac{1}{2}$. The bundle is stimulated if at least $(1 - \Delta)$ N lines of the bundle are stimulated. The bundle is unstimulated if at most ΔN lines of the bundle are unstimulated. Otherwise the bundle is in an uncertain state and the network malfunctions. The generalized element is designed to realize the same function as the basic element. In some cases the input bundles of a generalized element may each be stimulated or unstimulated while the output state is uncertain. To minimize this effect the author introduces, besides an "executive organ" intended to realize a given function, a "restoring organ" intended to transform (with high probability) an input bundle with stimulation level near zero (one) to an output bundle with stimulation level nearer zero (one). To perform this restorative function it must be assumed that the stimulated (unstimulated) lines are distributed at random in a bundle. It is not considered, in detail, how this randomness can be effected (and maintained). The author obtains detailed quantitative results for networks constructed out of a generalized Sheffer stroke element. If $\Delta = .07$ and $\epsilon < .0107$ (ϵ is the probability of error of a single line Sheffer stroke element independent of the state of the network) and if the generalized Sheffer stroke element is constructed out of the single line Sheffer elements, then the probability of malfunction of the network may be made arbitrarily small by taking N sufficiently large. In certain examples, rough calculations show that with $\epsilon = .005$, N must be about 20,000 to 25,000 to obtain "good" reliability.

In "Some Uneconomical Robots," J. T. Culbertson shows how to realize arbitrary truth functions by interconnecting neuron-like elements. He, apparently, regards his robots as "existing" for a finite amount of time only. Since he seeks (in this paper) to design only "uneconomical" robots he does not consider neuron nets with cycles.

M. L. Minsky in "Some Universal Elements for Finite Automata" is concerned with constructing automata, out of certain types of elements, which will

realize certain, rather restricted, kinds of behavior. A "stimulus" is a finite set of (input) states at certain discrete instants of time. A "response" is analogous. A "response function" is defined on a finite set of stimuli with values which are responses. An automaton "realizes" a response function, f, if whenever it receives a stimulus, s, which is in the domain of f, its response is $f(s)$. The author's main result is that a wide class of response functions (which he describes) can be realized (with some delay) by automata constructed from disjunctions, conjunctions, and any "non-monotonic" element. Inasmuch as the domain of the response functions is finite, his problem is similar to that of finding complete sets of primitives for truth functions. The problem is complicated by the fact that his primitives all have a "built in" unit delay.

M. D. Davis in "A Note on Universal Turing Machines" proposes a definition of "universal Turing machine." Let Z be a Turing machine. An instantaneous description of Z is a pair $<q, E>$ consisting of a state q of the reading head of Z, and an expression E with a marked place, composed of symbols of Z. Let δ_Z be the set of Gödel numbers of all instantaneous descriptions $<q, E>$ such that if E is printed on the tape of Z, and the reading head is put in state Q scanning the marked place of E, then Z will eventually halt. Now the machine Z is defined to be universal if the set δ_Z is recursively enumerable and has the further property that to every recursively enumerable set R there is a recursive function $\sigma(x)$ such that $R = \{x \mid \sigma(x) \epsilon\, \delta_Z\}$. Using methods of the theory of recursive functions (particularly Kleene's T-function) the author shows: 1. How an arbitrary machine Z can be so encoded on a universal machine U, as to make U carry out the computation performed by Z (this, of course, is relative to particular conventions concerning the use of machines to compute functions.) 2. That a Turing machine which would (relative to the adopted convention concerning computations) ordinarily be called universal (on more or less intuitive grounds) is universal in the precise sense of the definition. 3. That for a universal machine U there is a machine C which performs the encoding mentioned in 1, and which is not itself a universal machine. These results are intended to make it plausible that the given definition of universality is reasonable. In particular 3 shows that the encoding necessary to make a universal machine perform the computation of another machine is simple in a precise sense.

C. E. Shannon in "A Universal Turing Machine with Two Internal States" investigates the number of tape symbols and internal states required for universal Turing machines. If U is a Turing machine, let $st(U)$ denote the number of states which the reading head of U can take on, and let $sy(U)$ denote the number of symbols which the reading head can print on the tape. Given a machine A to construct a machine B which acts essentially like A and such that $st(B) < st(A)$ (or $sy(B) < sy(A)$). Part of this problem, of course, is to adopt reasonable interpretations of "essentially acts alike." The specific results obtained are: 1. To every machine A one can construct a machine B such that $st(B) = 2, sy(B) = sy(A) + 4sy(A) \cdot st(A)$, and such that B essentially acts like A in the following sense: Let $A_1, \cdots, A_n$ be the symbols of A, let $A_1, \cdots, A_n, B_1, \cdots, B_m$ be the symbols of B, let T_0 be any tape which carries symbols A_i only, and let c be a marked cell of T_0. By properly changing the symbols in c and one of the neighboring cells to symbols B_i one can obtain a tape T_0' such that if the cells c of T_0 and T_0' at time 1 are placed under the reading heads of A and B respectively, there will be a sequence $1 = t_1 < t_2 < t_3 \cdots$

of time instances such that for any i the tape of A at time i carries the same symbols in corresponding cells as the tape of B at time t_i, except for the cells under the reading heads and one of the neighboring cells. 2. There is a universal Turing machine U such that $st(U) = 2$. This is clearly a corollary to 1. It is shown also that a machine W with $st(W) = 1$ cannot be universal. 3. To every machine A one can construct a machine C such that $sy(C) = 2$ and $st(C) < 8 \cdot st(A) \cdot sy(A)$, and such that C acts essentially the same as A in roughly the sense that if the symbols of A are coded by use of the two symbols of C, then the machine C acts on the coded tape T' like the machine A acts on T. These results suggest that it is possible, within limits, to exchange symbols for states and vice versa, with about a factor 8 of change in the product $st(M) \cdot sy(M)$. It is conjectured that the loss factor 8 could be decreased if "essentially acts like" is interpreted as "having the same calculating abilities in the large" rather than as "behaving alike in a local sense."

K. de Leeuw, E. F. Moore, C. E. Shannon, and N. Shapiro in an article entitled "Computability by Probabilistic Machines" seek to answer the question: "Is there anything that can be done by a machine with a random element but not by a deterministic machine?" A machine is defined in such a way that if its input is a function f on the natural numbers to zero, one that is an infinite sequence of zeros and ones then, if f is recursive, its output may be interpreted as an enumeration of a recursively enumerable set of natural numbers. For f fixed, given a recursively enumerable set there exists a machine which will enumerate it. If the machine is now modified so as to receive as input a function f such that $f(n) = 1$ with probability p (independent of n), $0 < p < 1$, the machine is called a p-machine. A set is *strongly enumerated* by a p-machine if the probability that the set will occur as the output of the machine is non-zero. A set is *weakly enumerated* by a p-machine if the probability that x occurs in the output is greater than $\frac{1}{2}$ if and only if x is an element of the set. The main result of the paper may be stated as the equivalence of the following three statements concerning a set S:

1) S is enumerated by some machine whose input is the binary expansion of p.
2) S is strongly enumerated by some p-machine.
3) S is weakly enumerated by some p-machine.

From this it follows readily that if p is computable, i.e., the sequence representing p is recursive, then p-machines can only strongly (weakly) enumerate recursively enumerable sets while if p is not computable some p-machines will strongly (weakly) enumerate non-recursively enumerable sets. The authors define a *computable stochastic machine* (*c-s* machine) as an object having a countable (finite or denumerable) set of states, a distinguished initial state, a countable set of output symbols and an effective function which when applied to the sequence $x_1, x_2, \cdots, x_{n+1}, s, m$ $(n \geq 0)$ where the x's are states, s is an output symbol and m is a positive integer, yields the first m bits of the binary expansion of the probability that the machine will "next" be in state x_{n+1} and "print" s if it has, up to the "present," been in the states $x_1, x_2, \cdots, x_n$. The authors show that the *c-s* machines are equivalent (in a precisely defined sense) to p-machines with p-computable. From this result it follows that the outputs of *c-s* machines represent exactly the recursively enumerable sets.

J. McCarthy in "The Inversion of Functions Defined by Turing Machines"

considers the task of designing machines to solve well-defined intellectual problems. This task is identified with inverting functions defined by Turing machines. (The identification could be made plausible by referring to formal theories.) Suppose ϕ_m is the function calculated by the m-th Turing machine. Let $g(m, n)$ be called an inversion-function if (1) $f_m(g(m, n)) = n$ in case there exists an x such that $f_m(x) = n$, and (2) $g(m, n)$ is undefined otherwise. The problem is to construct machines which "efficiently" calculate values of an inversion-function. Suggestions are made as to what efficiency should be taken to mean. No theorems are proved.

The papers of W. Ross Ashby, "Design for an Intelligence Amplifier," and D. M. MacKay, "The Epistemological Problem for Automata," are parallel in many respects: i) each is concerned with synthesizing automata which can perform functions until now reserved to the mental processes of the higher animals, e.g., solving difficult problems or forming concepts, ii) each presents descriptions of specific mechanisms and then suggests ways of constructing a general device using principals embodied in the specific device, iii) each makes random fluctuations a basic part of the mechanism (although in different ways), iv) each makes comparison a central part of the process (again in different ways). Ashby's process requires that the universe of potential solutions to a given problem be specified either by means of a model or by letting the mechanism operate directly on the appropriate part of the real world. The mechanism then makes random selections from the potential solutions and compares them with the conditions to be satisfied. The basic unit of the process proposed by MacKay consists of a comparator, a "control-signed generated and mixer" M, and an effector. The comparator compares input (from outside the unit) with effector output and produces a signal which gives a measure of different between the two corresponding to the error signal in a servomechanism. This error signal is used to activate, via M, a new effector output. MacKay's process then consists of a hierarchy of such units with the output of M at one level acting as input to the next (as well as activating the effector at its own level). The input of the lowest level unit comes from the environment of the whole device. MacKay concludes: "... an automaton designed on statistical principles, which can evolve an internal organizing routine to respond adaptively to regularities of its sensory input, is capable in principle of developing its own symbols for concepts of any order of abstraction, including metalinguistic concepts, without prior instruction ..." MacKay's mechanism seems to be more sophisticated than the one proposed by Ashby in that it allows for learning. That is, Ashby's upon being given the same problem a second time can be expected to take as long as before; not so MacKay's. MacKay's mechanism when presented with a problem of searching for regularities in the input at any level will improve its performance after several runs in the sense of becoming more efficient at reducing the error signal.

In "Conditional Probability Machines," Albert M. Uttley discusses a machine whose response to stimuli are similar in some ways to those of animals and gives a set of sufficient design requirements for such a machine. The design incorporates the mathematical principles of set inclusion and conditional probability, i.e., Bayes' theorem. Each unit computes the probability of occurrence of the stimulus J which corresponds to it when another stimulus K occurs. This probability depends on the past history of joint occurrences of stimuli. "Temporal and Spatial

Patterns in a Conditional Probability Machine" discusses the problem of distinguishing between temporal patterns by a method of delayed inputs. The discussion on spatial patterns is rather vague.

J. R. Büchi
C. C. Elgot
J. H. Holland
R. A. Laing
R. F. Lyjak
J. B. Wright

The University of Michigan
Ann Arbor, Michigan

54[X].—Her Majesty's Nautical Almanac Office, *Interpolation and Allied Tables*, Her Majesty's Stationery Office, London, England, 1956, 80 p. 24.5 cm. Price 5 s. net.

This is a new edition of a booklet with the same title, first issued in 1936 as a reprint of certain pages of the British Nautical Almanac, and which was frequently reprinted, sometimes with additions. It derives largely from the experience, over several decades, of the late L. J. Comrie, Superintendent of H. M. Nautical Almanac Office, his successor, D. H. Sadler, and their disciples. In its present form it provides an excellent practical introduction to a large part of classical numerical analysis and can be thoroughly recommended as an inexpensive compact text, complete with tables. The typography is excellent, and the reviewer's only complaint concerns the redundant decimal points in Table 6.

I. Part I contains a section on elementary computing principles, which is followed by an account of the principles and practices of interpolation and some remarks on numerical differentiation and integration. There are worked examples showing the use of the tables in the second part.

II. Part II contains various tables of Bessel and Everett coefficients, some critical and some direct. To see the scope of these tables it will be worth while sketching formally current (British) practice in the interpolation. The latest developments, of applying the ideas of Chebyshev economization to the problems of interpolation are discussed briefly in Part III. They will be discussed briefly in a forthcoming review of the booklet of L. Fox [5] in this journal.

One method is to truncate the Bessel formula

$$f_p = f_0 + p\delta_{1/2} + B_2(\delta_0^2 + \delta_1^2) + B_3\delta_1^3 + B_4(\delta_0^4 + \delta_1^4) + \cdots$$

at an appropriate place. Here

$$B_2(p) = B_2(1 - p) = \tfrac{1}{2}p(p - 1)/2!,$$

$$B_3(p) = -B_3(1 - p) = p(p - 1)(p - \tfrac{1}{2})/3!,$$

$$B_4(p) = B_4(1 - p) = \tfrac{1}{2}(p + 1)p(p - 1)(p - 2)/4!.$$

The modified form of this, with fourth and sixth differences thrown back is

$$f_p = f_0 + p\delta_{1/2} + B_2(\delta_{m0}^2 + \delta_{m1}) + B_3\delta_1^3$$

or, if δ^4, δ^6 are varying slowly,

$$f_p = f_0 + p\delta_{1/2} + B_2(\delta_{m0} + \delta^2_{m1}) + B_3(\delta^2_{m1} - \delta^2_{m0})$$

where

$$\delta_m{}^2 = \delta^2 - 0.184\delta^4 + 0.038\delta^6.$$

Otherwise we use the Everett formula

$$f_p = (1 - p)f_0 + pf_1 + (E_2\delta_0{}^2 + F_2\delta_1{}^2) + (E_4\delta_0{}^4 + F_4\delta_1{}^4) + \cdots$$

truncated appropriately. Here

$$E_2(p) = F_2(1 - p) = -p(p - 1)(p - 2)/3!$$

$$E_4(p) = F_4(1 - p) = -(p + 1)p(p - 1)(p - 2)(p - 3)/5!.$$

Various modifications of this are possible. Corresponding to * we have

$$f_p = (1 - p)f_0 + pf_1 + (E_2\delta^2_{m0} + F_2\delta^2_{m1})$$

with δ^2_m defined as above. If we throw the sixth and eighth differences back into the fourth according to

$$\delta_m{}^4 = \delta^4 - 0.207\delta^6 + 0.045\delta^8,$$

keeping the second differences as they are, we have

$$f_p = (1 - p)f_0 + pf_1 + (E_2\delta_0{}^2 + F_2\delta_1{}^2) + (E_4\delta^4_{m0} + F_4\delta^1_{m1}).$$

We can also use a simultaneous throw-back of δ^6, δ^8, δ^{10} into δ^2, δ^4 according to

$$\delta_m{}^2 = \delta^2 - 0.01312\delta^6 + 0.0043\delta^8 - 0.001\delta^{10}$$

$$\delta_m{}^4 = \delta^4 - 0.27827\delta^6 + 0.0685\delta^8 - 0.016\delta^{10}$$

and use

** $$f_p = (1 - p)f_0 + pf_1 + (E_2\delta^2_{m0} + F_2\delta^1_{m1}) + (E_4\delta^4_{m0} + F_4\delta^4_{m1}).$$

Finally, we can use a generalized throw-back

$$\delta_m{}^2 = \delta^2 - 0.184\delta^4 + 0.038082\delta^6 - 0.00830\delta^8 + \cdots$$

$$10\gamma^4 = 0.01\delta^4 - 0.002783\delta^6 + 0.00068\delta^8 + \cdots$$

(here γ^4 is a modified fourth difference (divided by 1000)). With this notation we can use

$$f_p = (1 - p)f_0 + pf_1 + (E_2\delta^2_{m0} + F_2\delta^2_{m1}) + (M_4\gamma_0{}^4 + N_4\gamma_1{}^4)$$

or, if γ^4 is varying slowly

$$f_p = (1 - p)f_0 + pf_1 + (E_2\delta^2_{m0} + F_2\delta^2_{m1}) + T_4(\gamma_0{}^4 + \gamma_1{}^4)$$

where

$$M_4 = 1000(E_4 + 0.184E_2), \qquad N_4 = 1000(F_4 + 0.184F_2),$$

$$T_4 = \tfrac{1}{2}(M_4 + N_4).$$

In recent British tables, e.g., those of the Airy integral [1] and the Weber functions [2], the γ's are tabulated where necessary. Modified differences have been used in many tables since about 1930.

The twelve tables given, which we shall now describe in detail, are sufficient for most applications of these methods of interpolation.

1. *First-difference correction* $p\delta_{1/2}$. This gives to the nearest integer, the value of $p\delta_{1/2}$ for $p = .1(.1)9$ and $\delta_{1/2} = 1(1)128$. The table is arranged in blocks corresponding to eight consecutive values of $\delta_{1/2}$ and on each block is given the appropriate contribution of the second decimal of p, i.e., $p\delta_{1/2}$ for $p = .01(.01).09$ for an average $\delta_{1/2}$ (2 pages).

2. *Second-difference correction* $B_2(\delta_0^2 + \delta_1^2)$. This gives to the nearest integer the values of $-B_2(\delta_0^2 + \delta_1^2)$ for $p = 0(.01).5$ and for $\delta_0^2 + \delta_1^2 = 10(5)200$.

3. *Bessel coefficients* B_2, B_3, B_4. There is a critical table of B_2, proceeding in units of the third decimal, with p given to $4D$, together with $3D$ values of B_3 and B_4 at interval .1.

4. *Third-difference correction* $B_3\delta_{\frac{1}{2}}^3$. This gives to the nearest integer, the values of $B_3\delta_{\frac{1}{2}}^3$ for $p = 0(.01).5$ for $\delta_{\frac{1}{2}}^3 = 100(100)1000$.

5. *Fourth-difference correction* $B_4(\delta_0^4 + \delta_1^4)$. This gives to the nearest integer, the values of $B_4(\delta_0^4 + \delta_1^4)$ for $p = 0(.01).5$ and $\delta_0^4 + \delta_1^4 = 100(100)1900$.

6. *Second-difference coefficients* B_2, E_2, F_2. This is a critical table, proceeding in units of the fourth decimal; p is given to $5D$. Such a table is feasible since $2B_2 = E_2 + F_2$ (7 pages).

7. *Bessel coefficients* B_2, B_3, B_4. This gives B_2 to $6D$, B_3 to $5D$ and B_4 to $4D$ for $p = 0(.001).5$. First differences of B_2 are given (5 pages).

8. *Throw-backs* $0.184\ \delta^4$ *and* $0.038\delta^6$. This is a critical table, proceeding in units, with integral values of δ^4 and δ^6 up to $\delta^4 = 1002$, $\delta^6 = 250$.

9. *Throw-backs* $0.207\delta^6$ *and* $0.045\delta^8$. This is a critical table of $0.207\ \delta^6$ proceeding by 5 units up to 220, with integral values of δ^6, and also giving $0.207\delta^6$ to the nearest unit for $\delta^6 = 1000(1000)10000$. There is a critical table of $0.045\delta^8$ proceeding by 5 units up to 40, giving integral values of δ^8.

10. *Residual fourth-difference correction* $T_4(\gamma_0^4 + \gamma_1^4)$. This gives the values of the correction, to the nearest integer, for $p = 0(.02).5$ and $(\gamma_0^4 + \gamma_1^4) = .001(\delta_0^4 + \delta_1^4) = 2(2)24$.

11. *Short table of Bessel and Everett coefficients.* This gives B_ν for $\nu = 2\ (1)11$ and $E_{2\nu}$, $F_{2\nu}$ for $\nu = 1(1)5$, to $6D$ for $p = .1(.1).9$.

12. *Coefficients for Everett's formula.* This gives E_2, and F_2 to $7D$, M_4 and N_4 to $3D$, T_4 to $2D$, E_4 and F_4 to $6D$, E_6 and F_6 to $5D$, all for $p = 0(.001).5$ (10 pages).

III. Part III is a collection of relevant formulae, subdivided into eight sections, some of which, e.g., that on the solution of ordinary differential equations, contain much valuable commentary. We deal briefly with each section.

A. *Operators.* The finite difference operators are defined and their interrelations are recorded.

B. *Differences.* The notation is discussed and there are short tables useful in connection with checking by differencing.

C. *Interpolation.* In addition to the material summarized in our account of Part II (supplemented, of course, by working limits for applicability of the

various formulae) there are sections on the magnitude of the Bessel coefficient, on interpolation to halves, on the Lagrangian interpolation method and Aitken's scheme for carrying it out and on adjusted divided differences. There is a discussion of the central difference formulae of Stirling and Steffensen for $|p| \leq 1/2$. A polynomial form of ** is given and there is a discussion, with tables, of the use of economized polynomials for interpolation [3, 4, 5]. The problem of sub-tabulation is not discussed: it will be the subject of a companion booklet [6] (5 pages).

D. *Differentiation.* This includes expressions for the r-th derivative, $r=1(1)10$, evaluated at a tabular point, or at a halfway point, with coefficients given both as rational numbers and as decimals. Expressions in terms of ordinates and in terms of forward and backward differences (for use near the ends of a table) are also given, in addition to expressions for differences in terms of derivatives.

A table of coefficients of the differences in the expressions for the first three derivatives at a non-tabular point p is given, for $p=0(.01)5$. These expressions are obtained by differentiating the Bessel formula.

The formulae appropriate for tables with modified differences are also given (5 pages).

E. *Integration.* Various formulae for evaluating definite integrals in terms of various differences, derivatives, and ordinates are given. In particular, the coefficients (of the ordinates) for use in Gregory's formula, the corresponding central difference formula, and the Gaussian quadrature are given to 8, 8, and 10 decimals.

The case of simple and double integration to non-tabular points is discussed and suitable tables are provided. There is a selection of starter, predictor, and corrector formulae for use in the numerical integration of differential equations (5 pages).

F. *Differential equations.* This account is restricted to initial value problems for ordinary differential equations. Five methods are mentioned: (1) local Taylor series, (2) Runge-Kutta, (3) Adams-Bashforth, (4) Central difference extrapolation methods, (5) Deferred-correction method (5 pages).

G. *The precision of calculation.* This includes a table giving the absolute errors and the variances of errors resulting from certain arithmetical operations on quantities subject to error, e.g., n-th difference, or the result of a direct or inverse interpolation. There is a short table of $\phi(x) = e^{-x^2/2}/\sqrt{2\pi}$ and $P(t) = 200\int_t^\infty \phi(x)dx$, the percentage area of the tails of the normal curve.

H. *Constants.* A two-page table of constants to a reasonable number of places is given.

The booklet closes with a one-page bibliography.

JOHN TODD

California Institute of Technology
Pasadena, California

1. J. C. P. MILLER, *The Airy Integral*, Part-volume B of British Assoc. Adv. Sc., Math. Tables, Cambridge University Press, 1946.
2. NATIONAL PHYSICAL LABORATORY, *Tables of Weber Parabolic Cylinder Functions*, Her Majesty's Stationery Office, London, 1955.

3. C. W. CLENSHAW & F. W. J. OLVER, "The use of economized polynomials in mathematical tables," Camb. Phil. Soc., *Proc.*, v. 51, 1955, p. 614–628.
4. ZDENĚK KOPAL, *Numerical Analysis*, John Wiley and Sons, Inc., New York, 1955.
5. L. FOX, *The Use and Construction of Mathematical Tables*, National Physical Lab., Math. Tables, v. 1, Her Majesty's Stationery Office, London, 1956.
6. HER MAJESTY'S NAUTICAL ALMANAC OFFICE, *Subtabulation, A Companion Booklet to Interpolation and Allied Tables*, Her Majesty s Stationery Office, London, 1958.

55[X].—J. L. SYNGE, *The Hypercircle in Mathematical Physics: A Method for the Approximate Solution of Boundary Value Problems*, Cambridge Univ. Press, New York, 1957, xii + 424 p., 24 cm. Price $13.50.

There are, roughly speaking, two lines of approach to the problem of the numerical solution of partial differential equations: The methods of finite differences and the methods based on ideas of Functional Analysis or the Calculus of Variations. There exist many connections between the theories of the two procedures, but most of the work done in this field falls distinctly into one of those two categories.

The present book gives a complete and detailed account of a variational method developed, during the past ten years, by the author and several other mathematicians. The underlying ideas are connected with the names of Thomson, Ritz, Rayleigh, and Trefftz, but the terminology, the mode of the presentation and the numerical techniques are original contributions of the author and a group of more recent mathematicians.

Here is, in a nutshell, a description of the hypercircle method: The problem is the approximate numerical solution of a boundary value problem of elliptic type in a region V. A suitable Hilbert space F is introduced such that to every function in V there corresponds a point in F. If F is suitably defined the functions that satisfy the boundary conditions and the functions that satisfy the differential equation define two linear subspaces L' and L'' of F, respectively, that are orthogonal. They intersect at the point S corresponding to the solution of the boundary value problem. The position of S is unknown, but if one chooses r points of L' and s points of L'' one can determine, by linear combination, the hyperplane L'_r of dimension r in L' through the first r points and, similarly, a hyperplane L''_s in L'' of dimension s. It is a simple matter to calculate the end points V', V'' of the shortest straight segment joining L'_r and L''_s. If these $r + s$ points, i.e., the corresponding functions in the domain V, are skillfully chosen the segment $V'V''$ in Hilbert space will be short, and its midpoint C will be a good approximation to the true solution S. The error can be appraised by observing that S lies on the hypersphere with diameter $V'V''$, thanks to the orthogonality of L' and L''. Also, the $r + s$ known points of F limit the position of S to a known hyperplane defined by $r + s$ linear equations. The intersection of the hypersphere with this hyperplane is the hypercircle from which the book takes its name. Simple geometric arguments based on the triangle inequality lead then to convenient inclusion theorems by means of which the error in the Hilbert norm can be appraised. Pointwise bounds on the error are then obtainable from an inequality derived with the help of a fundamental solution of the differential equation.

The larger part of the book is devoted to applications based on the technique of so-called pyramid functions. These are functions that are piecewise linear in V with certain discontinuities admitted along straight lines (or planes, in three di-

mensions). The author shows how approximating subspaces L'_r, L''_s can be constructed by linear combination of such pyramid functions. Approximation by piecewise linear functions is a natural bridge between the variational and the finite difference methods. This aspect, which in the reviewer's opinion deserves further study, is only briefly discussed in the book.

This book is intended for engineers and physicists. The author is aware of the fact that many readers in this group shy away from abstract arguments in Functional Analysis. He therefore develops his topic very slowly and carefully assuming no more mathematical knowledge than a good course in Advanced Calculus will impart. The systematic appeal to geometric intuition assisted by a large number of diagrams, as well as many numerical examples that are described in great detail, is intended to make the subject palatable to the applied scientist. On the other hand, the proofs for all statements are mathematically complete.

The author is a skillful expositor and writes in a very attractive, lively style. Whether he has steered the safest course between the Scylla of excessive mathematical conciseness and the Charybdis of tiresome repetitiousness in his explanations is hard for this reviewer to decide. A book of 424 pages seems a rather lengthy account of one particular numerical method. It is true that the reader will learn much that is of more general importance, such as the most elementary properties of Hilbert space, the simpler aspects of the theory of elliptic differential equations, and some elements of the theory of elasticity with particular emphasis on the torsion problem.

The examples are, with one exception, worked out with the help of desk calculators only. The question as to the suitability of this method for automatic computing machines is, unfortunately, not discussed.

WOLFGANG WASOW

Mathematics Research Center
U. S. Army
University of Wisconsin
Madison, Wisconsin

56[X].—J. KUNTZMANN, "Evaluations d'Erreur dans les Representations Approchees de Derivees," Societé d'Électronique et d'Automatisme, 138 Boulevard de Verdun, Courbevoie (Seine), France, 1955, 49 p. (mimeographed), 30.5 cm.

These tables serve to supplement an earlier report [1] by the same author.

The error $R_{p,q,n}$ in the numerical differentiation formulas

$$f^{(p)}(x_q) = \sum_{i=0}^{n} A_{i,p,q,n} y_i + R_{p,q,n},$$

is given by

$$R_{p,q,n} = \int_{-\infty}^{\infty} K_{p,q}(t) f^{(n+1)}(t)\, dt$$

where

$$K_{p,q}(t) = \sum_{i=0}^{n} \epsilon_i A_{i,p,q,n} \frac{(x_i - t)^n}{n!}, \qquad \epsilon_i = \begin{matrix} 1 \text{ if } t \text{ is between } x_i \text{ and } x_q, \\ 0 \text{ otherwise.} \end{matrix}$$

The upper bounds for the remainder terms

$$P_{p,q,n}h^{n+1}\to M_{n+1}$$

where

$$|f^{(n+1)}(t)| < M_{n+1}$$

and the quantities

$$H_{p,q,n} = \int_{-\infty}^{+\infty} |K_{p,q,n}(t)|\,dt$$

are tabulated for $n = 1(1)11$, p, $q = 0(1)n - 1$.

M. ABRAMOWITZ

National Bureau of Standards
Washington, D. C.

1. J. KUNTZMANN, "Formules de Dérivation Approchée au Moyen de Points Équidistants," Report No. 1.373/1, Societé d'Électronique et d'Automatisme, 138 Boulevard de Verdun, Courbevoie (Seine), France, 1954. (MTAC, Rev. 51, v. 10, 1956, p. 171–172.)

57[Z].—PRESTON C. HAMMER, Editor, *The Computing Laboratory in the University*, The University of Wisconsin Press, Wisconsin, 1957, xv + 236 p. 23 cm. Price $6.50.

A set of 31 papers dealing with various aspects of the computing laboratory in the University. These papers, each presumably presenting the individual views of its author, were presented at a conference held by the University of Wisconsin in August 1955. The papers vary in style and in content, but in total they convey an idea of many of the problems, many of the pleasures and some of the expenses of running a computing laboratory in a University.

The University of Wisconsin should be thanked for sponsoring this conference and for making the material presented available in printed form.

The table of contents follows:

Computing and the University. The computing laboratory in the University by C. A. Elvehjem; The new significance of computation in higher education by J. H. Curtiss; Equipmental aids to computing by Jay W. Forrester.

Applications of Computing in Science. Weather prediction by Philip Duncan Thompson; Computing in astronomy by W. J. Eckert; Applications of computing to fluid dynamics problems by Harwood G. Kolsky; Applications of high-speed computing to chemical problems by Joseph O. Hirschfelder; The impact of fast computers on physics by Marshall Rosenbluth; The use of desk calculators by Paul S. Dwyer.

Applications of Computing in Industry. The computer laboratory in industry by H. R. J. Grosch; Applications of computing in the Aircraft Industry by H. S. Wolanski; Computers improve power system performance by L. K. Kirchmayer; Assignment, programing, and scheduling by David F. Votaw, Jr.

Future Demands for Personnel in Computing. Future demands for trained personnel by E. K. Ritter; Supply and demand in computational mathematics by Forman S. Acton; The future demand for mathematicians in the computing field by R. E. Gaskell; Future demands for engineers and scientists in the field of computtation by Eldred C. Nelson.

Curriculums for Computing. The contribution of the computing laboratory to the University curriculum by Charles W. Adams; The educational program in Numerical Analysis of the Department of Mathematics, U.C.L.A. by George E. Forsythe; Curriculum needs in the computing field by Vincent C. Rideout; The Numerical Analysis Program at the University of Maryland by David M. Young, Jr.

Equipping a University Computing Laboratory. Equipping the University computation laboratory by John W. Carr III; Equipping a University computing laboratory by C. C. Gotlieb; Equipping a University laboratory to satisfy the computational demands by H. O. Hartley; Equipping a University computing laboratory by Ralph E. Meagher; Equipping the university computing laboratory by Alan J. Perlis.

Organizing and Financing a University Computing Laboratory. Organizing and financing a University computing laboratory by J. P. Nash; On organizing and financing a laboratory by Carl F. Kossack; The University computation laboratory as a cooperative industry-education project by Arvid W. Jacobson; The Cornell Computing Center, a minimum University computing laboratory by R. J. Walker.

Computation in Perspective. Dangerous gulfs: some reflections on the social implications of computing machines by J. H. Van Vleck.

C. B. T.

58[Z].—*Proceedings of the Third Annual Computer Applications Symposium*, held October 9–10, 1956, sponsored by the Armour Research Foundation of the Illinois Institute of Technology, Chicago, Illinois, 1957, 148 p., 23 cm. Price $3.00.

This conference had as its theme the application of medium scale digital computers. It is stated that "a primary purpose of the symposium was to illustrate diversity among computing machines and among applications, to provide a true picture of medium-scale computer applications as existing today." Following a description of the medium scale computers to be considered and introductory articles, "The election and the UNIVAC" by C. Collingwood, and "Model making problems in electric forecasting" by M. A. Woodbury, there are two sections of papers, one devoted to Business and Management Applications and the other to Engineering and Research Applications. The table of contents of these two sections follows:

Business and Management Applications

Development of a Products Pipe Line Simulator on an NCR 102A by J. H. Mallas

Application of the IBM 650 to Stock Brokerage Operations by V. Lazzaro

The Elecom 125 in Personnel Classification Research by J. M. Leiman

Programming Ordinary Life Insurance Operations for the Datatron by J. S. Hill

Manufacturing Data Processing on the IBM 650 by H. H. Marlow, Jr.

The Univac File-Computer Applied to General Accounting Functions by T. R. Lyon

Panel Discussion—Business and Management Applications
Engineering and Research Applications

Solution of Rotating Electric Machinery Problems with ALWAY by C. G. Veinott

The IBM 650 Applied to Problems of the Electrical Industry by R. Haberman, Jr. and F. J. Maginniss
The NCR 102A as an Aid in Training and Research by E. J. Stewart
Optical Calculations using the Burroughs E101 by A. Cox
Use of the Datatron in the Petroleum Industry by J. S. Aronofsky
Panel Discussion—Engineering and Research Applications

C. B. T.

59[Z].—R. HUNT BROWN, *Office Automation*, Automation Consultants, Inc., New York, 1955, looseleaf, xviii + 283 p. Price $12.50. "Service to keep manual up-to-date" offered at $25.00 per year.

This book starts with a readable section on the "what," "why," and "how," of office automation, goes on to extensive discussions of hardware, especially electronic computers, briefly considers integrated-data-processing accounting systems, and closes with short sections on sociological implications, scientific frameworks, and probable future developments related to office automation. The manner of presentation is aimed at the nontechnician, but computer experts not widely familiar with business applications will find much of interest in this book.

The core of *Office Automation* seems to this reviewer to be the "Hardware Section" (p. 25–204), which indeed comprises over 60 per cent of the book's total wordage, under chapter headings which can be condensed as follows: The common language concept; Integrated data processing; Electrical communication facilities; Common language machines; Native language machines; Tabulating and computing equipment; Punched card machine accounting; Fundamentals of electronic computers (four chapters); Principal electronic business computers; Memory systems; Output printers; Programming; Computer capabilities; Choosing a computer; Rent or buy; Miscellaneous topics (three chapters). The greatest value of this "Hardware Section" lies in its compact but expressive descriptions of specific equipment. If the promised up-dating service is well maintained, this book will continue to be a handy reference for everyone who is concerned with commercial computing equipment.

One criticism, of a type which can only too often be levelled at semipopular treatments, is that too much of the material reads like a "sales-pitch." For example, on p. 8, integrated data processing and electronic equipment is described as "the answer" (for the present purpose it matters not to what question). However, the "show-me" attitude of the typical business reader probably demands such a treatment, and almost surely guarantees that little or no harm will result from it.

On the positive side, having discussed concepts of automation with several business administration students before and after they read *Office Automation*, the reviewer feels that it does get the main points across, and does so effectively and without undue effort on the reader's part.

JAMES R. JACKSON

University of California,
Los Angeles

TABLE ERRATA

The following erratum is mentioned in this issue:

J. PETERS, *Ten-Place Logarithms of the Numbers from* 1 *to* 100,000 Review 1, page —.

259.—MATHEMATICAL TABLES PROJECT, New York, A. N. LOWAN, technical director, *Tables of Lagrangian Interpolation Coefficients*, Columbia University Press, New York, 1944 and 1948 editions. [*MTAC*, Rev. 162, v. 1, 1943–1945, p. 314–315.]

Page xxx in Introduction—*In formulas* (*12*), (*13*), (*12a*), and (*13a*), *the factor* $1/n!$ *should be deleted.*

GERTRUDE BLANCH

214 W. Beechwood Avenue
Dayton 5, Ohio

260.—I. M. VINOGRADOV, *Elements of Number Theory*, Translated from the fifth edition by SAUL KRAVETZ, Dover Publications, Inc., New York, 1954.

Four of the five errata in the English translation [1] occur in this edition. The prime 2753 is listed correctly. In [1] the first erratum should read:

$p = 71$. *For* ind $57 = 43$, *read* ind $57 = 42$.

ANDRZEJ MAKOWSKI

Warszawa 10
ul. Bagatela 15 m. 49
POLAND

1. *MTAC*, Rev. 59, v. 9, 1955, p. 125.

NOTES

Research Potential and Training in the Mathematical Sciences

The University of Chicago has published results of a survey of research potential and training in the mathematical sciences undertaken under grants to that University by the National Science Foundation. The Chairman of the committee in charge was A. Adrian Albert, and the Investigator was John W. Green. Leon W. Cohen represented the National Science Foundation. The investigation included visits (and a few letters) to principal mathematics departments in American universities and colleges and questionnaires submitted to a large majority of the holders of the Ph.D. degree in mathematics. The report contains tabulations of some of the more interesting information gathered covering the opinions of many mathematicians concerning the ease with which mathematical research can be carried out under present conditions and their opinions of changes which might be made to improve these conditions.

Some undistributed copies of the report remain and may be available to libraries of properly interested organizations from the Program Director for Mathematics, The National Science Foundation, Washington 25, D. C.

C. B. T.